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# Some results on vanishing moments of wavelet packets, convolution and cross-correlation of wavelets

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**Abstract.** A formula for calculating moments for wavelet packets is derived and a sufficient condition for moments of wavelet packets to be vanishing is obtained. Also, the convolution and cross-correlation theorems for Hilbert transform of wavelets are proved. Finally, using MRA of $L^2(\mathbb{R})$, some results on the vanishing moments of the scaling functions, wavelets and their convolution in two dimension are given.

Keywords: Wavelet packets; Hilbert transformation; Moments; Wavelet packets

Mathematics Subject Classification: 42C40; 44A15; 44A60; 65T60

## 1. INTRODUCTION

In 1984, the combined effort of Grossmann and Morlet [7] directed to a complete mathematical study of the continuous wavelet transforms and their various applications. In 1988, the concept of Multiresolution Analysis (MRA) was introduced by S. Mallat [16] and Y. Meyer [17]. Using MRA, wavelet spaces are constructed by splitting the frequency domain dyadically and their bases are obtained with the help of translated and dialated form of a

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single function. A stronger extension of wavelets and MRA is wavelet packets which are particularly the superposition of wavelets and are especially well adapted for signal processing. In 1988, Daubechies [4] found a new method to construct the compactly supported orthogonal wavelet. A major problem at that time was to deal with the poor frequency localization of wavelet bases and the solution was proposed by Coifman et al. [3] in 1990 as a result of which they introduced the notion of wavelet packets which ensured better frequency localization for the bases and thereby provided more adequate decomposition containing stationary and transient components. They retain many of the significant characteristics such as smoothness, orthogonality and localization properties of their root wavelets.

In 2005, Soares et al. [20] observed that if $\psi(t)$ is a real wavelet, then Hilbert transform of $\psi(t)$ is also a real wavelet with same energy and admissibility coefficient of its generating wavelet. Later in 2009, Chaudhury and Unser [2] observed that the fundamental reasons why the Hilbert transform can be seamlessly integrated into the multiresolution framework of wavelets are its scale and translation invariances, and its energy-preserving (unitary) nature. For various details related to Hilbert transform one may refer to [6,14]. In 2015, Khanna et al. [9,10] studied vanishing moments of Hilbert transform of wavelets and proved certain results to approximate the functions in $L^2(\mathbb{R})$. Later, in 2016, Khanna et al. [11] introduced the orthogonal Coifman wavelet packet systems, the biorthogonal Coifman wavelet packet systems, and also introduced the notion of Hilbert transform of wavelet packets, and Hartley-like wavelet packets. Recently, in 2017, Khanna et al. [12] studied vanishing moments of wavelet packets and define the wavelets associated with Riesz projectors. Very recently, Khanna et al. [13] studied wavelet packets and give various results related to their moments.

In this paper, moments of wavelet packets have been calculated and a sufficient condition under which wavelet packets have vanishing moments is given. Hilbert transform wavelet convolution and Hilbert transform wavelet cross-correlation theorems have been given to analyze the Hilbert transform of convolved and cross-correlated functions (or signals). Further, we develop a relationship between the vanishing moments of wavelets and the Hilbert transform of convolved (or cross-correlated) wavelets. Finally, some results on the vanishing moments of the scaling function, wavelets, and their convolutions in two dimensions have been given.

## 2. Preliminaries

Recall from [8,15] that a sequence of closed subspaces $(V_j)_{j\in\mathbb{Z}}$, of $L^2(\mathbb{R})$ is called a multiresolution analysis (MRA), if

(i) $V_j \subset V_{j+1}$, for all $j \in \mathbb{Z}$;
(ii) $f \in V_j$ if and only if $f(2(\cdot)) \in V_{j+1}$, for all $j \in \mathbb{Z}$;
(iii) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$;
(iv) $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R})$;
(v) There exists a function $\phi \in V_0$, such that $\{\phi(\cdot - k) : k \in \mathbb{Z}\}$ is an orthonormal basis for $V_0$.

The function $\phi$ whose existence is asserted in (v) is called a scaling function of the given MRA.

Let $M_{2,p}$ be the uniquely defined Daubechies wavelet matrix of rank 2 and genus $p$ given by $M_{2,p} = \begin{bmatrix} r_0 & \cdots & r_{2p-1} \\ s_0 & \cdots & s_{2p-1} \end{bmatrix}$. We define $r_k = 0, s_k = 0$, for $k \notin [0, 2p-1]$.

The Daubechies scaling function $\phi$ and wavelet function $\psi$ of genus $p$ satisfy the usual scaling equation $\phi(x) = \sum_{k=0}^{2p-1} r_k\ \phi(2x-k)$, for all $x \in \mathbb{R}$ and the wavelet equation $\psi(x) = \sum_{k=0}^{2p-1} s_k\ \phi(2x-k)$, for all $x \in \mathbb{R}$. For details, see [22]. Also, $\phi$ satisfies the normalization condition $\int_{\mathbb{R}} \phi(x)\,dx = 1$ and $\sum_{k\in\mathbb{Z}} \phi(x-k) = 1$, for all $x \in \mathbb{R}$. Wavelet packets were basically prompted to enhance the frequency of resolution of signals attained by wavelet analysis. The basic wavelet packets [19], $\omega_n, n = 0, 1, 2, \ldots,$ are defined by the recursion formulae given as $\omega_{2n}(x) = \sum_{k=0}^{2p-1} r_k\ \omega_n(2x-k),\ \omega_{2n+1}(x) = \sum_{k=0}^{2p-1} s_k\ \omega_n(2x-k)$ or equivalently, in terms of the Fourier transform, we have $\widehat{\omega}_{2n}(\eta) = m_0(\frac{\eta}{2})\ \widehat{\omega}_n(\frac{\eta}{2})$ and $\widehat{\omega}_{2n+1}(\eta) = m_1(\frac{\eta}{2})\ \widehat{\omega}_n(\frac{\eta}{2})$, where the symbols $m_0$ and $m_1$ are associated with the above sequences by $m_0(\eta) = \sum_{k=0}^{2p-1} r_k e^{\iota k\eta}$ and $m_1(\eta) = \sum_{k=0}^{2p-1} s_k e^{\iota k\eta} = e^{\iota\eta}\overline{m_0(\eta+\pi)}$.

Also, if we write $n \in \mathbb{N}$ into its unique dyadic expansion $n = \sum_{j=1}^{\infty} \epsilon_j 2^{j-1}, \epsilon_j \in \{0, 1\}$, we have a general expression of the Fourier transform of the basic wavelet packets given by

$$\widehat{\omega}_n(\eta) = \prod_{j=1}^{q} m_{\epsilon_j}(2^{-j}\eta)\ \widehat{\omega}_0(2^{-q}\eta), \text{ where } q = max\{j : \epsilon_j = 1\}. \tag{2.1}$$

## 3. Vanishing moments of wavelet packets

We begin this section with the following definition of vanishing moments given in [10]. A function $f(x)$ is said to have $k$ *vanishing moments* if $\int_{\mathbb{R}} x^v f(x)\,dx = 0,\ 0 \le v \le k-1$, where $\int_{\mathbb{R}} x^v\ f(x)\,dx$ is known as the $v$th moment of $f(x)$, denoted by $Mom_v(f)$.

In the following result, we give a formula for calculating the moments of wavelet packets.

**Theorem 3.1.** *The moments of wavelet packets $\omega_n$ are given by*

$$Mom_v(\omega_{2n}) = \sum_{\sum_{1\le t\le q+2} f_t = v} \Bigg\{ {}^vC_{f_1,f_2,\ldots,f_{q+2}} \frac{i^v}{2^{f_1+2f_2+\cdots+(q+1)(f_{q+1}+f_{q+2})}}\ m_0^{(f_1)}(0) \left(\prod_{1\le t\le q} m_{\epsilon_t}^{(f_{t+1})}(0)\right)\left(\frac{i^{f_{q+2}}}{2^{f_{q+2}}-1}\ m_{f_{q+2}} + \sum_{l=1}^{f_{q+2}-1} d_l Mom_l(\omega_0)\right)\Bigg\},$$

$$Mom_v(\omega_{2n+1}) = \sum_{\sum_{1\le t\le q+2} f_t = v} \Bigg\{ {}^vC_{f_1,f_2,\ldots,f_{q+2}} \frac{i^v}{2^{f_1+2f_2+\cdots+(q+1)(f_{q+1}+f_{q+2})}}\ m_1^{(f_1)}(0) \left(\prod_{1\le t\le q} m_{\epsilon_t}^{(f_{t+1})}(0)\right)\left(\frac{i^{f_{q+2}}}{2^{f_{q+2}}-1}\ m_{f_{q+2}} + \sum_{l=1}^{f_{q+2}-1} d_l Mom_l(\omega_0)\right)\Bigg\},$$

*where* $d_l = {}^{f_{q+2}}C_l \frac{(-i)^l}{(2^{f_{q+2}}-1)}\ m_0^{(f_{q+2}-l)}(0)$ *and* $m_{f_{q+2}} = \sum_{k=0}^{2p-1}(r_k\ k^{f_{q+2}})$ *and* ${}^vC_{f_1,f_2,\ldots,f_{q+2}} = \frac{v!}{f_1!f_2!\ldots f_{q+2}!}$ $(v \in \mathbb{N})$ *are the multinomial coefficients. The sum is taken over all combinations of nonnegative integer indices* $f_1, \ldots, f_{q+2}$ *such that the sum of all* $f_i$*'s is* $v$.

**Proof.** Consider $\widehat{\omega}_{2n}(\eta) = m_0\left(\frac{\eta}{2}\right)\ \widehat{\omega}_n\left(\frac{\eta}{2}\right)$. Using (2.1), we obtain

$$\widehat{\omega}_{2n}(\eta) = m_0\left(\frac{\eta}{2}\right) \prod_{j=1}^{q}\left(m_{\epsilon_j}(2^{-(j+1)}\eta)\right)\ \widehat{\omega}_0(2^{-(q+1)}\eta).$$

Now, $Mom_v(\omega_{2n}) = \widehat{t^v\, \omega_{2n}(t)}(0)$. Therefore

$$\begin{aligned} Mom_v(\omega_{2n}) &= i^v\, \widehat{\omega_{2n}}^{(v)}(0) \\ &= i^v \left[ m_0\Big(\frac{\eta}{2}\Big) \Big(\prod_{j=1}^{q} m_{\epsilon_j}(2^{-(j+1)}\eta)\Big)\, \widehat{\omega}_0(2^{-(q+1)}\eta)\right]^{(v)} \\ &= \sum_{\sum_{1\le t\le q+2} f_t = v} \Big\{ {}^{v}C_{f_1,f_2,\ldots,f_{q+2}} \frac{i^v}{2^{f_1+2f_2+\cdots+(q+1)(f_{q+1}+f_{q+2})}}\, m_0^{(f_1)}(0) \\ &\qquad \Big(\prod_{1\le t\le q} m_{\epsilon_t}^{(f_{t+1})}(0)\Big) \times \widehat{\omega}_0^{(f_{q+2})}(0)\Big\}, \end{aligned}$$

where ${}^{v}C_{f_1,f_2,\ldots,f_{q+2}} = \frac{v!}{f_1!f_2!\ldots f_{q+2}!}$, $v \in \mathbb{N}$ denotes the multinomial coefficients and the sum is taken over all combinations of nonnegative integer indices $f_1, \ldots, f_{q+2}$ such that the sum of all $f_i$'s is $v$. This further gives

$$\begin{aligned} Mom_v(\omega_{2n}) = \sum_{\sum_{1\le t\le q+2} f_t = v} &\Big\{ {}^{v}C_{f_1,f_2,\ldots,f_{q+2}} \frac{i^v}{2^{f_1+2f_2+\cdots+(q+1)(f_{q+1}+f_{q+2})}}\, m_0^{(f_1)}(0) \\ &\Big(\prod_{1\le t\le q} m_{\epsilon_t}^{(f_{t+1})}(0)\Big)\Big(\frac{1}{(2^{f_{q+2}}-1)} \sum_{l=1}^{f_{q+2}} \Big({}^{f_{q+2}}C_l\, m_0^{(l)}(0)\, \widehat{\omega_0}^{(f_{q+2}-l)}(0)\Big)\Big)\Big\}. \end{aligned}$$

Since $m_0^{(l)}(0) = \sum_{k=0}^{2p-1} r_k\, (ik)^l$ and $\hat{\omega}_0^{f_{q+2}-l}(0) = Mom_{f_{q+2}-l}(\omega_0)\, (-i)^{f_{q+2}-l}$, it follows that

$$\begin{aligned} Mom_v(\omega_{2n}) &= \sum_{\sum_{1\le t\le q+2} f_t = v} \Big\{ {}^{v}C_{f_1,f_2,\ldots,f_{q+2}} \frac{i^v}{2^{f_1+2f_2+\cdots+(q+1)(f_{q+1}+f_{q+2})}}\, m_0^{(f_1)}(0) \\ &\qquad \Big(\prod_{1\le t\le q} m_{\epsilon_t}^{(f_{t+1})}(0)\Big)\Big(\frac{1}{(2^{f_{q+2}}-1)} \sum_{l=1}^{f_{q+2}} \Big({}^{f_{q+2}}C_l \sum_{k=0}^{2p-1} (r_k\, (ik)^l) \\ &\qquad Mom_{f_{q+2}-l}(\omega_0)\, (-i)^{f_{q+2}-l}\Big)\Big)\Big\} \\ &= \sum_{\sum_{1\le t\le q+2} f_t = v} \Big[ {}^{v}C_{f_1,f_2,\ldots,f_{q+2}} \frac{i^v}{2^{f_1+2f_2+\cdots+(q+1)(f_{q+1}+f_{q+2})}}\, m_0^{(f_1)}(0) \\ &\qquad \Big(\prod_{1\le t\le q} m_{\epsilon_t}^{(f_{t+1})}(0)\Big)\Big(\frac{1}{(2^{f_{q+2}}-1)}\Big\{ \sum_{k=0}^{2p-1} \Big(r_k\, (ik)^{f_{q+2}}\, Mom_0(\omega_0)\Big) \\ &\qquad + \sum_{l=1}^{f_{q+2}-1} \Big({}^{f_{q+2}}C_l\, m_0^{(l)}(0)\, \widehat{\omega_0}^{(f_{q+2}-l)}(0)\Big)\Big\}\Big)\Big] \\ &= \sum_{\sum_{1\le t\le q+2} f_t = v} \Big\{ {}^{v}C_{f_1,f_2,\ldots,f_{q+2}} \frac{i^v}{2^{f_1+2f_2+\cdots+(q+1)(f_{q+1}+f_{q+2})}}\, m_0^{(f_1)}(0) \\ &\qquad \Big(\prod_{1\le t\le q} m_{\epsilon_t}^{(f_{t+1})}(0)\Big)\Big(\frac{i^{f_{q+2}}}{(2^{f_{q+2}}-1)} m_{f_{q+2}} + \sum_{l=1}^{f_{q+2}-1} d_l\, Mom_l(\omega_0)\Big)\Big\}, \end{aligned}$$

where $d_l = {}^{f_{q+2}}C_l \frac{(-i)^l}{(2^{f_{q+2}}-1)}\, m_0^{(f_{q+2}-l)}(0)$ and $m_{f_{q+2}} = \sum_{k=0}^{2p-1}(r_k\, k^{f_{q+2}})$.

Similarly, we have

$$Mom_v(\omega_{2n+1}) = \sum_{\sum_{1\le t\le q+2} f_t = v} \left\{ {}^{v}C_{f_1,f_2,\ldots,f_{q+2}} \frac{i^v}{2^{f_1+2f_2+\cdots+(q+1)(f_{q+1}+f_{q+2})}} m_1^{(f_1)}(0) \left( \prod_{1\le t\le q} m_{\epsilon_t}^{(f_{t+1})}(0) \right) \times \left( \frac{i^{f_{q+2}}}{2^{f_{q+2}}-1} m_{f_{q+2}} + \sum_{l=1}^{f_{q+2}-1} d_l Mom_l(\omega_0) \right) \right\},$$

where $d_l = {}^{f_{q+2}}C_l \frac{(-i)^l}{(2^{f_{q+2}}-1)} m_0^{(f_{q+2}-l)}(0)$ and $m_{f_{q+2}} = \sum_{k=0}^{2p-1}(r_k\, k^{f_{q+2}})$. □

In the following result, we give a sufficient condition for moments of wavelet packets to be vanishing.

**Theorem 3.2.** *The wavelet packet moments, $Mom_v(\omega_n), n \neq 0$ vanishes for $v = 0, 1, 2, \ldots, (p-1)$ if for a wavelet matrix $M_{2,p}$, each of the following conditions is satisfied.*

(a) $m_{f_{q+2}} = 0$, *i.e., $(f_{q+2})^{th}$ moment of scaling parameters $r_k$ vanishes,*

(b) $Mom_l(\omega_0)$ *vanishes for $l = 1, 2, \ldots, (f_{q+2}-1)$,*

*where $1 \le f_{q+2} \le v$ and the sum of all combinations of non-negative integer indices $f_1, \ldots, f_{q+2}$ is $v$.*

**Proof.** The proof can be worked out on the lines of Theorem 3.1. □

## 4. Hilbert transform of wavelets

Recall from [14] that the *Hilbert transform* of a function $f$ on a real line is defined by

$$\mathcal{H}f(x) = \frac{1}{\pi} lim_{\epsilon\to 0} \int_{|x-t|\ge\epsilon} \frac{f(t)}{x-t} dt = \frac{1}{\pi} lim_{\epsilon\to 0} \int_{|t|\ge\epsilon} \frac{f(x-t)}{t} dt,$$

provided that the limit exists in some sense.

Also, the *moment formula* for the Hilbert transform of $f$ is given by

$$\mathcal{H}\{x^n\ f(x)\} = x^n \mathcal{H}f(x) - \frac{1}{\pi} \sum_{m=0}^{n-1} x^m \int_{\mathbb{R}} z^{n-1-m} f(z)\, dz, n \ge 0.$$

Note that the above formula holds if $x^n f(x) \in L^p(\mathbb{R})$, $1 < p < \infty$.

In the following results, we show that the wavelet transform of Hilbert transform of convolved (cross-correlated) signals with Hilbert transform of convolved (cross-correlated) wavelets can be decomposed as the convolution (cross-correlation) of the wavelet transform of Hilbert transform of a signal with a wavelet and the wavelet transform of the other signal with Hilbert transform of other wavelet.

**Theorem 4.1** (*Hilbert Transform Wavelet Convolution Theorem*)**.** *Let $\psi_1, \psi_2$ be two wavelets such that $\psi_1, \hat{\psi}_1 \in L^1(\mathbb{R})$ and $\psi_2 \in L^2(\mathbb{R})$ and let $W_{\psi_1} g'$ and $W_{\psi_2'} h$ be the continuous wavelet transform of two functions $g' = \mathcal{H}g$ and $h$ with wavelets $\psi_1$ and $\psi_2' = \mathcal{H}\psi_2$, respectively, where $g \in L^2(\mathbb{R})$ and $h, \hat{h} \in L^1(\mathbb{R})$. If $f = g * h$ and $\psi = \psi_1 * \psi_2$, then*

$$W_{\psi'} f'(a, b) = \frac{1}{|a|^{\frac{1}{2}}} (W_{\psi_1} g' * W_{\psi_2'} h)(a, b),$$

*where $f'$ and $\psi'$ denotes the Hilbert transform of $f$ and $\psi$, respectively and '$*$' denotes a convolution operator.*

**Proof.** The continuous wavelet transform of $f'$ with respect to $\psi'$ may be written as

$$W_{\psi'} f'(a,b) = \frac{1}{|a|^{\frac{1}{2}}} \int_{\mathbb{R}} f'(x)\, \overline{\psi'}\left(\frac{x-b}{a}\right) dx$$

$$= \frac{1}{|a|^{\frac{1}{2}}} \int_{\mathbb{R}} \int_{\mathbb{R}} g'(t)\, h(x-t)\, dt \int_{\mathbb{R}} \overline{\psi_1}(y)\, \overline{\psi_2{}'}\left(\frac{x-b}{a} - y\right) dy\, dx.$$

Writing $x - t = p$ and $b + ay - t = q$, we have

$$W_{\psi'} f'(a,b) = \frac{1}{|a|^{\frac{3}{2}}} \int_{\mathbb{R}} \int_{\mathbb{R}} g'(t)\, \overline{\psi_1}\left(\frac{t-(b-q)}{a}\right) dt \int_{\mathbb{R}} h(p)\, \overline{\psi_2{}'}\left(\frac{p-q}{a}\right) dp\, dq.$$

$$= \frac{1}{|a|^{\frac{1}{2}}} \int_{\mathbb{R}} W_{\psi_1} g'(a, b-q)\, W_{\psi_2'} h(a,q)\, dq$$

$$= \frac{1}{|a|^{\frac{1}{2}}} (W_{\psi_1} g' * W_{\psi_2'} h)(a,b)$$

which is the convolution of wavelet transform of Hilbert transform of a signal $g$ with a wavelet $\psi_1$ and the wavelet transform of the other signal $h$ with Hilbert transform of other wavelet $\psi_2$. □

**Theorem 4.2** (*Hilbert Transform Wavelet Cross-Correlation Theorem*)**.** *Let $\psi_1, \psi_2$ be two wavelets such that $\psi_1, \hat{\psi}_1 \in L^1(\mathbb{R})$ and $\psi_2 \in L^2(\mathbb{R})$ and let $W_{\psi_1} g'$ and $W_{\psi_2'} h$ be the continuous wavelet transform of two functions $g' = \mathcal{H}g$ and $h$ with wavelets $\psi_1$ and $\psi_2' = \mathcal{H}\psi_2$, respectively, where $g \in L^2(\mathbb{R})$ and $h, \hat{h} \in L^1(\mathbb{R})$. If $f = g \otimes h$ and $\psi = \psi_1 \otimes \psi_2$, then*

$$W_{\psi'} f'(a,b) = \frac{1}{|a|^{\frac{1}{2}}} (W_{\psi_1} g'(a,b) \otimes W_{\psi_2'} h(a,-b)),$$

*where $f'$ and $\psi'$ denotes the Hilbert transform of $f$ and $\psi$, respectively and '$\otimes$' denotes a cross-correlation operator.*

**Proof.** The continuous wavelet transform of $f'$ with respect to $\psi'$ may be written as

$$W_{\psi'} f'(a,b) = \frac{1}{|a|^{\frac{1}{2}}} \int_{\mathbb{R}} f'(x)\, \overline{\psi'}\left(\frac{x-b}{a}\right) dx$$

$$= \frac{1}{|a|^{\frac{1}{2}}} \int_{\mathbb{R}} \int_{\mathbb{R}} g'(t)\, \overline{h}(t+x)\, dt \int_{\mathbb{R}} \overline{\psi_1}(y)\, \psi_2{}'\left(y + \frac{x-b}{a}\right) dy\, dx.$$

Writing $t + x = p$ and $b + t - ay = q$, we have

$$W_{\psi'} f'(a,b) = -\frac{1}{|a|^{\frac{3}{2}}} \int_{\mathbb{R}} \int_{\mathbb{R}} g'(t)\, \overline{\psi_1}\left(\frac{t-(q-b)}{a}\right) dt \int_{\mathbb{R}} \overline{h}(p)\, \psi_2{}'\left(\frac{p-q}{a}\right) dp\, dq$$

$$= \frac{1}{|a|^{\frac{1}{2}}} \int_{\mathbb{R}} W_{\psi_1} g'(a, q-b)\, \overline{W}_{\psi_2'} h(a,q)\, dq,$$

$$= \frac{1}{|a|^{\frac{1}{2}}} (W_{\psi_1} g' \otimes W_{\psi_2'} h(a,-b))$$

which is the cross-correlation of wavelet transform of Hilbert transform of a signal $g$ with a wavelet $\psi_1$ and the wavelet transform of the other signal $h$ with Hilbert transform of other wavelet $\psi_2$. $\square$

Next, we prove that the number of vanishing moments of the Hilbert transform of convolved (cross-correlated) wavelets is the sum of the number of vanishing moments of wavelets involved.

**Theorem 4.3.** *Let $\psi_1, \psi_2 \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ be two wavelets with $m_1$ and $m_2$ vanishing moments, respectively and let $\psi_3 = (\psi_1 * \psi_2)$ be the convolution of the wavelets $\psi_1$ and $\psi_2$. Then, $\psi_3{}' = \mathcal{H}\psi_3$ has $m_1 + m_2$ vanishing moments provided that $t^{m_2}\psi_2(t) \in L^2(\mathbb{R})$.*

**Proof.** Since $\psi_3{}'$ is an admissible wavelet, it follows that

$$\begin{aligned}\int_{\mathbb{R}} t^r\ \psi_3{}'(t)\,dt &= \int_{\mathbb{R}} t^r\ (\psi_1 * \psi_2{}')(t)\,dt\\ &= \int_{\mathbb{R}} \psi_1(x)\,dx \int_{\mathbb{R}} t^r\ \psi_2{}'(t-x)\,dt.\end{aligned}$$

Writing $t - x = z$, we have

$$\begin{aligned}\int_{\mathbb{R}} t^r\ \psi_3{}'(t)\,dt &= \sum_{n=0}^{r} {}^rC_n \int_{\mathbb{R}} x^n\ \psi_1(x)\,dx \int_{\mathbb{R}} z^{r-n}\psi_2{}'(z)\,dz\\ &= \sum_{n=0}^{r} {}^rC_n\ Mom_n(\psi_1)\ Mom_{r-n}(\psi_2{}').\end{aligned}$$

Also, $t^{m_2}\psi_2(t) \in L^2(\mathbb{R})$. Therefore, using moment formula for the Hilbert transform, we have

$$\int_{\mathbb{R}} t^n \psi_2'(t)\,dt = 0 \text{ for } 0 \le n \le m_2.$$

Let $r \le m_1 + m_2 - 1$. If $r - n \le m_2$, then $Mom_{r-n}(\psi_2{}') = 0$, otherwise $n \le m_1 - 1$ which gives $Mom_n(\psi_1) = 0$.
Hence the number of vanishing moments of $\psi_3{}'(t)$ is $m_1 + m_2$. $\square$

**Theorem 4.4.** *Let $\psi_1, \psi_2 \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ be two wavelets with $m_1$ and $m_2$ vanishing moments, respectively and let $\psi_3 = (\psi_1 \otimes \psi_2)$ be the convolution of the wavelets $\psi_1$ and $\psi_2$. Then, $\psi_3{}' = \mathcal{H}\psi_3$ has $m_1 + m_2$ vanishing moments provided that $t^{m_2}\psi_2(t) \in L^2(\mathbb{R})$.*

**Proof.** The proof can be worked out on the lines of Theorem 4.3. $\square$

## 5. Moments of two dimensional wavelets

Consider two-dimensional spaces $\boldsymbol{V}_j$, $j \in \mathbb{Z}$ as the tensor product of two one dimensional multiresolution analyses $V_j$, $j \in \mathbb{Z}$. Define $\boldsymbol{V}_j$, $j \in \mathbb{Z}$ by

$$\boldsymbol{V}_0 = V_0 \otimes V_0 = \overline{\text{span}\,\{\boldsymbol{U}(x, y) = u(x)v(y) : u, v \in V_0\}}.$$

Then, $\boldsymbol{V}_j$ forms a multiresolution analysis (MRA) of $L^2(\mathbb{R}^2)$ satisfying

(i) $... \subset \boldsymbol{V}_{-2} \subset \boldsymbol{V}_{-1} \subset \boldsymbol{V}_0 \subset \boldsymbol{V}_1 \subset \boldsymbol{V}_2 \subset \boldsymbol{V}_2 \subset ...$,

(ii) $\bigcap_{j\in\mathbb{Z}} \boldsymbol{V}_j = \{(0,0)\}$, $\overline{\bigcup_{j\in\mathbb{Z}} \boldsymbol{V}_j} = L^2(\mathbb{R}^2)$,

(iii) $\boldsymbol{U} \in \boldsymbol{V}_0 \Leftrightarrow \boldsymbol{U}(2^j\cdot, 2^j\cdot) \in \boldsymbol{V}_{j+1}$.

(iv) The set $\{\Phi_{0,k_1,k_2}(\cdot,\cdot) : k_1, k_2 \in \mathbb{Z}\}$ forms an orthonormal basis for $\boldsymbol{V}_0$, where $\Phi_{j,k_1,k_2}(x,y) = 2^j\ \Phi(2^j x - k_1, 2^j y - k_2) = 2^j\ \phi(2^j x - k_1)\,\phi(2^j y - k_2),\ j, k_1, k_2 \in \mathbb{Z}$.

For each $j \in \mathbb{Z}$, the complement space $\boldsymbol{W}_j$ is the orthogonal complement of $\boldsymbol{V}_j$ in $\boldsymbol{V}_{j+1}$ such that

$$\begin{aligned}
\boldsymbol{V}_{j+1} &= V_{j+1} \otimes V_{j+1} \\
&= (V_j \oplus W_j) \otimes (V_j \oplus W_j) \\
&= (V_j \otimes V_j) \oplus [(W_j \otimes V_j) \oplus (V_j \otimes W_j) \oplus (W_j \otimes W_j)] \\
&= \boldsymbol{V}_j \oplus \boldsymbol{W}_j.
\end{aligned}$$

The space $\boldsymbol{W}_j$ called the "detail space" is itself made up of three orthogonal subspaces which leads us to define three two-dimensional wavelets $\Psi^1(x,y) = \phi(x)\ \psi(y)$, $\Psi^2(x,y) = \psi(x)\ \phi(y)$ and $\Psi^3(x,y) = \psi(x)\ \psi(y)$. Then, $\{\Psi^m_{j,k_1,k_2} : k_1, k_2 \in \mathbb{Z},\ m = 1, 2 \text{ or } 3\}$ is an orthonormal basis for $\boldsymbol{W}_j$ and $\{\Psi^m_{j,k_1,k_2} : j, k_1, k_2 \in \mathbb{Z},\ m = 1, 2 \text{ or } 3\}$ is an orthonormal basis for $\overline{\oplus_{j\in\mathbb{Z}} \boldsymbol{W}_j} = L^2(\mathbb{R}^2)$, where $\Psi^m_{j,k_1,k_2}(x,y) = 2^j\ \Psi^m_{j,k_1,k_2}(2^j x - k_1, 2^j x - k_2)$. For details see [1,5].

In the following result, we find the number of vanishing moments for two-dimensional wavelets.

**Theorem 5.1.** *Let $\phi$ be an orthogonal scaling function with $m_1$ vanishing moments and $\psi$ be the corresponding wavelet with $m_2$ vanishing moments. Then, the number of vanishing moments of two dimensional scaling function $\Phi$ and the associated two-dimensional wavelet $\Psi^3$ are $2m_1 - 1$ and $2m_2 - 1$ respectively, whereas the number of vanishing moments of the associated two-dimensional wavelets $\Psi^m$ for m = 1, 2 is $m_1 + m_2 - 1$.*

**Proof.** Note that

$$\begin{aligned}
\int_{\mathbb{R}}\int_{\mathbb{R}} x^p\ y^q\ \Psi^1(x,y)\,dx\,dy &= \int_{\mathbb{R}} x^p\ \phi(x)\,dx \int_{\mathbb{R}} y^q\ \psi(y)\,dy \\
&= Mom_p(\phi)\ Mom_{N-p}(\psi),\ \text{ where } p + q = N.
\end{aligned}$$

Let $N \le m_1 + m_2 - 2$. If $p \le m_1 - 1$, then $Mom_p(\phi) = 0$. If $p \ge m_1 - 1$, then $N - p \le m_2 - 1$. Thus $Mom_{N-p}(\psi) = 0$. Therefore, we have

$$Mom_p(\phi) = Mom_q(\psi) = 0,\ \text{ for all } p + q \le m_1 + m_2 - 2.$$

Hence the number of vanishing moments of $\Psi^1(x,y)$ is given by $m_1 + m_2 - 1$.
Similarly, one can prove that the number of vanishing moments for $\Phi(x,y)$, $\Psi^2(x,y)$ and $\Psi^3(x,y)$ can be evaluated as $2m_1 - 1$, $m_1 + m_2 - 1$ and $2m_2 - 1$ respectively. □

In the following two results, we give sufficient conditions for the two-dimensional scaling function to be vanishing.

**Theorem 5.2.** *Let $\phi$ be an orthogonal scaling function having compact support and let the first three moments of the wavelet $\psi$ vanishes. Let $\Phi(x, y) = \phi(x)\phi(y)$ be a two dimensional scaling function. Then $Mom_{1,q}(T_k \Phi(x, y))$ and $Mom_{p,1}(T_k \Phi(x, y))$, $0 \leq p, q \leq n, n \in \mathbb{N}$ vanishes if $Mom_1(\phi) = -k$, where $k \in \mathbb{Z}$.*

**Proof.** Note that

$$
\begin{aligned}
Mom_{1,q}(T_k \Phi(x, y)) &= \int_{\mathbb{R}} x \,\mathbf{T}_k\phi(x)\, dx \int_{\mathbb{R}} y^q \,\mathbf{T}_k\phi(y)\, dy \\
&= (k\, Mom_0(\phi) + Mom_1(\phi)) \left(\sum_{i=0}^{q} {}^qC_i\, k^{q-i}\, Mom_i(\phi)\right).
\end{aligned}
$$

Thus, for $0 \leq q \leq n,\ n \in \mathbb{N}$, $Mom_{1,q}(\mathbf{T}_k \Phi(x, y))$ vanishes if $Mom_1(\phi) = -k$, where $k \in \mathbb{Z}$. A similar argument can be given for $Mom_{p,1}(\mathbf{T}_k \Phi(x, y))$, $0 \leq p \leq n,\ n \in \mathbb{N}$. □

**Theorem 5.3.** *Let $\phi$ be an orthogonal scaling function having compact support and suppose that the first three moments of the wavelet $\psi$ vanishes. If $\Phi(x, y) = \phi(x)\phi(y)$ be the two dimensional scaling function then $Mom_{2,q}(\mathbf{T}_k \Phi(x, y))$ and $Mom_{p,2}(\mathbf{T}_k \Phi(x, y))$, $0 \leq p, q \leq n, n \in \mathbb{N}$ vanishes if $Mom_1(\phi) = -k$, where $k \in \mathbb{Z}$.*

**Proof.** We have

$$
\begin{aligned}
\int_{\mathbb{R}}\int_{\mathbb{R}} x^2\, y^q\, \mathbf{T}_k \Phi(x, y)\, dx\, dy &= \int_{\mathbb{R}} x^2\, \mathbf{T}_k\phi(x)\, dx \int_{\mathbb{R}} y^q\, \mathbf{T}_k\phi(y)\, dy \\
&= (Mom_2(\phi) + 2k Mom_1(\phi) + k^2\, Mom_0(\phi)) \\
&\quad \times (\sum_{i=0}^{q} {}^qC_i\, k^{q-i}\, Mom_i(\phi)).
\end{aligned}
$$

In view of Theorem 1 in [21], $(Mom_1(\phi))^2 = Mom_2(\phi)$. Thus, $Mom_{2,q}(\mathbf{T}_k \Phi(x, y))$ vanishes if $Mom_1(\phi) = -k$, where $k \in \mathbb{Z}$.
A similar argument can be given for $Mom_{p,2}(\mathbf{T}_k \Phi(x, y))$, $0 \leq p \leq n,\ n \in \mathbb{N}$. □

Finally, we prove a result related to the number of vanishing moments of the convolution of two wavelets in $L^2(\mathbb{R}^2)$.

**Theorem 5.4.** *Let $\Psi_1(x, y) = \psi_1(x)\, \psi_1(y)$ and $\Psi_2(x, y) = \psi_2(x)\, \psi_2(y)$ be two admissible wavelets in $L^2(\mathbb{R}^2)$, where $\psi_1, \psi_2 \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ with $M_1$ and $M_2$ vanishing moments, respectively. Let $\Psi_3(x, y) = \Psi_1(x, y) * \Psi_2(x, y)$. Then $\Psi_3(x, y)$ is an admissible wavelet and has $2(M_1 + M_2) - 1$ vanishing moments.*

**Proof.** Note that

$$
\begin{aligned}
\Psi_3(x, y) &= \Psi_1(x, y) * \Psi_2(x, y) \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \Psi_1(t_1, t_2)\, \Psi_2(x - t_1, y - t_2) dt_1\, dt_2 \\
&= (\psi_1 * \psi_2)(x)\, (\psi_1 * \psi_2)(y).
\end{aligned}
$$

Also, we have

$$\begin{aligned} C_{\Psi_3} &= (2\pi)^2 \int_{\mathbb{R}^2} \frac{|\widehat{\Psi_3}(\gamma)|^2}{|\gamma|}\, d\gamma \\ &\le (2\pi)^2 \int_{\mathbb{R}} \frac{|\widehat{\psi_1}(\gamma_1)\, \widehat{\psi_2}(\gamma_1)|^2}{|\gamma_1|}\, d\gamma_1 \int_{\mathbb{R}} \frac{|\widehat{\psi_1}(\gamma_2)\, \widehat{\psi_2}(\gamma_2)|^2}{|\gamma_2|}\, d\gamma_2. \end{aligned}$$

Since $\psi_1 \in L^1(\mathbb{R})$, $\widehat{\psi_1}$ is a bounded function. So, there exists a positive real number $K$ such that $|\widehat{\psi_1}(\alpha)| \le K$, for all $\alpha \in \mathbb{R}$. This gives

$$\begin{aligned} C_{\Psi_3} &\le (2\pi K^2)^2 \int_{\mathbb{R}} \frac{|\widehat{\psi_2}(\gamma_1)|^2}{|\gamma_1|}\, d\gamma_1 \int_{\mathbb{R}} \frac{|\widehat{\psi_2}(\gamma_2)|^2}{|\gamma_2|}\, d\gamma_2 \\ &= (2\pi K^2)^2\, C_{\psi_2}^2 < \infty. \end{aligned}$$

Now, we calculate the moments of $\Phi_3$. Note that

$$\begin{aligned} \int_{\mathbb{R}^2} x^p\, y^q\, \Psi_3(x,y)\, dx\, dy &= \int_{\mathbb{R}} \int_{\mathbb{R}} x^p\, y^q\, \Psi_3(x,y)\, dx\, dy \\ &= Mom_p(\psi_1 * \psi_2)\, Mom_q(\psi_1 * \psi_2) \\ &= Mom_p(\psi_1 * \psi_2)\, Mom_{N-p}(\psi_1 * \psi_2), \end{aligned}$$

where $p+q = N$. In view of Theorem 1 in [18], $(\psi_1 * \psi_2)$ has $(M_1 + M_2)$ vanishing moments. If $p \le (M_1 + M_2 - 1)$, then $Mom_p(\psi_1 * \psi_2) = 0$. If not, then $N - p \le M_1 + M_2 - 1$. Thus $Mom_{N-p}(\psi_1 * \psi_2) = 0$. Therefore, $Mom_p(\psi_1 * \psi_2) = Mom_q(\psi_1 * \psi_2) = 0$, for all $p + q \le 2(M_1 + M_2 - 1)$.
Hence the number of vanishing moments of $\Psi_3(x, y)$ is given by $2(M_1 + M_2) - 1$. □

## Conclusion

We have seen that wavelets are usually designed with higher vanishing moments which make them orthogonal to the low degree polynomials and therefore, they have the ability to compress non-oscillatory functions. The smoother is wavelet $\psi$, the greater is the number of vanishing moments. For any wavelet family, vanishing moments are necessary for the smoothness of the wavelet functions. With the development of the formula for calculating the number of vanishing moments for wavelet packets, we can thereby enhance our working in this direction furthermore. Also, since convolution (cross-correlation) of two wavelets meet the required regularity and admissibility conditions, we can use them to examine the Hilbert transform of convolved and cross-correlated signals with the help of Hilbert transform wavelet convolution (cross-correlation) theorems which have not been studied earlier. Further, we develop a relationship between the vanishing moments of wavelets and the Hilbert transform of convolved (cross-correlated) wavelets and with the knowledge of vanishing moments for two dimensional wavelets, one may reinforce the scope of study in this field of signals.

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