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Metric dimension of generalized wheels

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Abstract. In a graph G, a vertex $w \in V(G)$ resolves a pair of vertices $u, v \in V(G)$ if $d(u, w) \neq d(v, w)$. A resolving set of G is a set of vertices S such that every pair of distinct vertices in $V(G)$ is resolved by some vertex in S. The minimum cardinality among all the resolving sets of G is called the metric dimension of G, denoted by $\beta(G)$. The metric dimension of a wheel has been obtained in an earlier paper (Shanmukha et al., 2002). In this paper, the metric dimension of the family of generalized wheels is obtained. Further, few properties of the metric dimension of the corona product of graphs have been discussed and some relations between the metric dimension of a graph and its generalized corona product are established.

Keywords: Resolving set; Metric dimension; Generalized wheel; Corona product

Mathematics Subject Classification: 05C56; 05C12

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1. Introduction

Throughout this paper, we consider graphs that are simple, finite, undirected and connected. Given a graph G, a vertex $w \in V(G)$ resolves a pair of vertices $u, v \in V(G)$ if $d(u, w) \neq d(v, w)$. A set $S \subseteq V(G)$ is said to be a resolving set of G, if every pair of distinct vertices of G is resolved by some vertex in S. In other words, a resolving set of G is a set of vertices $S = \{w_1, w_2, \ldots, w_k\}$ of G such that for each $u \in V(G)$, the vector $r(u|S) = (d(u, w_1), d(u, w_2), \ldots, d(u, w_k))$ uniquely identifies u. The k-vector $r(u|S)$ is called a metric code or S-location or S-code of $u \in V(G)$.

The minimum cardinality among all the resolving sets of a graph G is called the metric dimension of G, denoted $\beta(G)$. Further, a resolving set with minimum cardinality is called a metric basis and its elements are called landmarks.

The concept of metric dimension was introduced by F. Harary & R. A. Melter [5] and independently by P.J. Slater [9] under the name locating set. Since then, this parameter has been widely studied and has found applications in various real world problems pertaining to network discovery, robot navigation and pharmaceutical chemistry. In particular, results on the metric dimension of the cartesian product of finite and infinite graphs have been obtained in [2]. Some results involving the metric dimension of a graph and its total graph have been discussed in [14]. The metric dimension of some regular graphs such as the circulant graphs $C(n, \pm\{1, 2, 3, 4\})$ and hexagonal cellular networks has been obtained in [4,8]. The k-metric dimension of graphs has been discussed in [1,11–13]. In this paper, we obtain the metric dimension of the family of generalized wheels. We also study a few properties of the metric dimension of the corona product of graphs and obtain some results on the metric dimension of the generalized corona product of graphs.

We begin with defining some standard graphs and graph classes central to the paper.

Definition 1.1. A graph having its vertex set V and edge set E with $|V| = n$ and $E = \emptyset$ is said to be a totally disconnected graph, denoted $\overline{K}_n$. In particular, if $n = 1$, the graph is said to be trivial.

Definition 1.2. The generalized wheel, denoted by $W_{m,n}$, is a graph obtained by joining the vertices of $\overline{K}_m$ to every vertex of a cycle C_n. That is $W_{m,n} = C_n + \overline{K}_m$. The m vertices of $\overline{K}_m$ and the n vertices of C_n in $W_{m,n}$ are respectively called central vertices and rim vertices of $W_{m,n}$.

The generalized wheel $W_{3,6} = C_6 + \overline{K}_3$ is illustrated in Fig. 1.

Let G be a connected graph of order n and H be any arbitrary graph. The corona product of G and H is the graph obtained from G and H by taking one copy of G and n copies of H and joining by an edge the ith vertex of G to every vertex of H_i, the ith copy of H, $1 \leq i \leq n$, and is defined as follows.

Definition 1.3. Let G and H be two given graphs with $V(G) = \{v_1, v_2, \ldots, v_n\}$. Then the *corona product* $G \odot H$ is defined as;

$$V(G \odot H) = V(G) \bigcup \left(\bigcup_{i=1}^{n} V(H_i) \right)$$

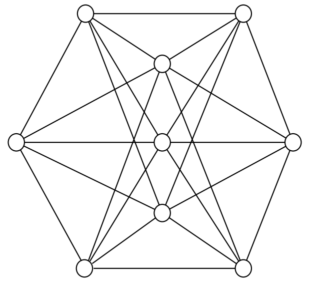

Fig. 1. The generalized wheel $W_{3,6} = C_6 + \overline{K}_3$.

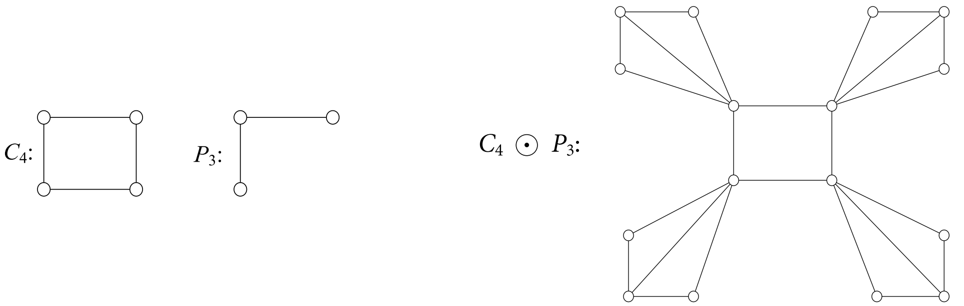

Fig. 2. The graphs C_4, P_3 and the corona product $C_4 \odot P_3$.

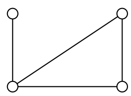

Fig. 3. The graph G.

$$E(G \odot H) = E(G) \bigcup \left(\bigcup_{i=1}^{n} \left[E(H_i) \cup \{v_i u_j : u_j \in V(H_i)\} \right] \right)$$

where $H_i \cong H$ for all $i = 1, 2, \ldots, n$.

The corona product $C_4 \odot P_3$ of the graphs C_4 and P_3 is illustrated in Fig. 2.

Definition 1.4. Let G be a connected graph of order n and H be a graph having n ordered components. Then, the graph called *generalized corona product* of G and H, denoted by $G \odot' H$, is obtained by superimposing ith vertex of G with a vertex of maximum degree in the ith component of H.

The generalized corona product $G \odot' H$ of the graphs G and H in Figs. 3 and 4 is illustrated in Fig. 5.

Observation 1.5. *By the definition of generalized corona product of graphs, we observe the following.*

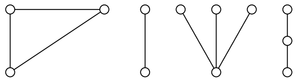

Fig. 4. The graph H having 4 components.

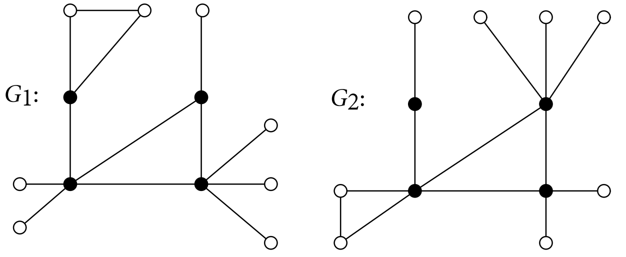

Fig. 5. Generalized corona products of graphs G and H, components of H assigned in two different orders.

1. *Let G be a connected graph of order n. Then $G \odot' H$ is defined only if H has exactly n components.*
2. *Let G be a graph of order n and $H_1, H_2, \ldots, H_n$ be the components of graph H. Then the structure of $G \odot' H$ depends on order in which the vertices of G and components of H are assigned.*
3. *Both G and H are subgraphs of $G \odot' H$.*
4. *Let p, q be order and size of G, p' and q' be order and size of H respectively. Then $G \odot' H$ is a connected graph of order $p' - p$ and size $q + q'$.*
5. *Both $G \odot' H$ and $H \odot' G$ exist if and only if G and H are trivial graphs.*
6. *$G \odot' H$ is isomorphic to G if and only if H is a totally disconnected graph (i.e., $H = \overline{K}_n$).*
7. *$G \odot' H$ is isomorphic to H if and only if G is a trivial graph.*

Remark 1.6. If $G \odot' H \cong G$ or $G \odot' H \cong H$, then the product is called trivial product.

Definition 1.7. In the generalized corona product, if each component of H is isomorphic to $G_i + K_1$, where G_i is any graph, then the product $G \odot' H$ is called the *super corona product* of G and H and is denoted by $G \hat{\odot} H$.

Remark 1.8. For every i, j, $1 \leq i, j \leq n$, if $G_i \cong G_j$, then the super corona product $G \hat{\odot} H$ is the usual corona product of graphs denoted by $G \odot G_i$.

2. Some known results on metric dimension

In this section, we recall some of the earlier works on metric dimension for immediate reference in the next and subsequent sections of the paper.

Theorem 2.1 (*S. Khuller et al.* [6]). *For a simple connected graph G, $\beta(G) = 1$ if and only if $G \cong P_n$.*

Theorem 2.2 (*S. Khuller et al. [6]*). *A graph G with $\beta(G) = 2$ cannot have K_5 as a subgraph.*

Theorem 2.3 (*B. Sooryanarayana [10]*). *For a graph G, let $G - me$ denote the graph, obtained from G, by deleting m arbitrary edges from G.Then, a graph G with $\beta(G) = k$ cannot have a subgraph isomorphic to $K_{2^k+1} - (2^{k-1} - 1)e$.*

Theorem 2.4 (*S. Khuller et al. [6]*). *A graph G with $\beta(G) = 2$ cannot have $K_{3,3}$ as a subgraph.*

Theorem 2.5 (*S. Khuller et al. [6]*). *Let G be a graph with metric dimension 2 and let $\{a, b\} \subset V(G)$ be a metric basis in G. The following are true:*

1. *There is a unique shortest path P between a and b.*
2. *The degrees of a and b are atmost 3.*
3. *Every other node on P has degree atmost 5.*

Theorem 2.6 (*S. Khuller et al. [6]*). *Let $G = (V, E)$ be a graph of order n with diameter d and metric dimension k. Then $|V| \leq d^k + k$.*

Theorem 2.7 (*F. Harary et al. [5]*). *For any positive integer n, $\beta(G) = n - 1$ if and only if $G \cong K_n$.*

Theorem 2.8 (*G. Chartrand et al. [3]*). *If G is a connected graph of order n, then $\beta(G) \leq n - diam(G)$.*

In view of Theorems 2.1 and 2.8, we have the following.

Lemma 2.9. *For any connected graph G on n vertices which is not a path,*

$$2 \leq \beta(G) \leq n - diam(G).$$

Theorem 2.10 (*B. Shanmukha et al. [7]*). *The metric basis of a wheel $W_{1,n}$, $n \geq 3$ cannot include the central vertex whenever $n \neq 3, 6$.*

Theorem 2.11 (*B. Shanmukha et al. [7]*). *For a wheel $W_{1,n}, n \geq 3$,*

1. $\beta(W_{1,3}) = \beta(W_{1,6}) = 3.$
2. $\beta(W_{1,4}) = \beta(W_{1,5}) = 2.$
3. $\beta(W_{1,x+5k}) = \begin{cases} 3 + 2k, & when\ x = 7\ or\ 8,\ k = 0, 1, 2 \ldots \\ 4 + 2k, & when\ x = 9\ or\ 10\ or\ 11,\ k = 0, 1, 2 \ldots. \end{cases}$

Observation 2.12. *Let $C = \{v_1, v_2, \ldots, v_m\}$ be the central vertices and $R = \{u_1, u_2, \ldots, u_n\}$ be the rim vertices of $W_{m,n}$. Then*

1. $d(v_i, v_j) = 2, \quad for \ \ every \ \ i \neq j,$
2. $d(u_i, u_j) = 2, \quad for \ \ every \ \ i \ \ and \ \ j \neq i + 1,$
3. $d(v_i, u_j) = 1, \quad for \ \ every \ \ i \ \ and \ \ j,$
4. $d(u_i, u_{i+1}) = 1, \quad for \ \ every \ \ i.$

3. Metric dimension of generalized wheels

In this section, we study some properties of the resolving set of generalized wheels using which we determine their metric dimension.

Lemma 3.1. *Let C and R respectively denote the set of central vertices and rim vertices of a Generalized Wheel $G = W_{m,n}$. Then for any resolving set S of G, $S \cap C \neq \phi$ whenever $m \geq 2$.*

Proof. If possible, suppose that $m \geq 2$ and S is a resolving set of $G = W_{m,n}$ such that $S \cap C = \emptyset$. Then $S \subseteq R$ and hence for the vertices $v_i, v_j \in C$ (exists since $|C| = m \geq 2$), we get $d(x, v_i) = d(x, v_j) = 1$ for all $x \in S$, a contradiction to the fact that S is a resolving set. □

Lemma 3.2. *Let C be the set of central vertices and S be a resolving set of the graph $W_{m,n}$. Then $|C - S| < 2$.*

Proof. Suppose S is a resolving set of $W_{m,n}$ and $|C - S| \geq 2$. Then there exist distinct vertices $v_l, v_j \in C$ not in S. Further, $d(v_l, x) = d(v_j, x) = 1$ for every $x \in S - C$ and $d(v_l, x) = d(v_j, x) = 2$ for every $x \in S \cap C$, Now, as $x \in S \Rightarrow x \in (S - C) \cup (S \cap C)$, we get $d(v_l, x) = d(v_j, x)$ for every $x \in S$, a contradiction. □

Lemma 3.3. *Let S be a resolving set and C be the set of central vertices of $W_{m,n}$ such that $C \subseteq S$. Then S is not a metric basis for $W_{m,n}$.*

Proof. Let S be a resolving set of $G = W_{m,n}$ which satisfies the hypothesis of the lemma. Let $v_k \in C$ be arbitrary and S' be the set $S' = S - \{v_k\}$. Suppose S is a metric basis of G. Then, S is a minimal resolving set with minimum cardinality so that by Lemma 2.9 $|S'| \leq n - 2$, that is, there exists $u, v \in V(G) - S'$. Further, as S is a resolving set, there is a vertex $w \in S$ such that $d(u, w) \neq d(v, w)$. Now, if $w \neq v_k$ for every pair of vertices in $V(G) - S'$, then S' is a resolving set smaller than S, a contradiction to the minimality of S. Thus, at least one pair $u, v \in V(G) - S'$ is resolved only by $v_k \in C \subseteq S$. However, in such a case, we have the following implications;

I1: Exactly one of u, v is a rim vertex and the other a central vertex (since if $u, v \notin C$ or $u, v \in C$, then as $v_k \in C$, we get $d(v_k, u) = d(v_k, v) \Rightarrow v_k$ will not resolve u and v). Without loss of generality, we take $v \in C \subseteq S$ and $u \notin C$. But then, $v \in V(G) - S' = (V(G) - S) \cup \{v_k\} \subseteq (V(G) - C) \cup \{v_k\}$ (since $C \subseteq S$) implies that $v = v_k$.

I2: $|C| \geq 2$. Otherwise $|C| = 1$, a contradiction to the hypothesis that $C \subseteq S$ by Theorem 2.10. So, there exists $v_l \in C$ with $l \neq k$. Now, as $C \subseteq S$, $v_l \in S$ and $d(v_l, v_k) = 2$, $d(v_k, u) = 1$ which implies that the pair u, v_k is resolved by a vertex in S, a contradiction to the above implication that v_k only resolves u and $v(= v_k)$ in S.

Hence the set S' is a resolving set containing lesser elements than in S, a contradiction to the minimality of S. □

Using Lemmas 3.1–3.3, we conclude the following.

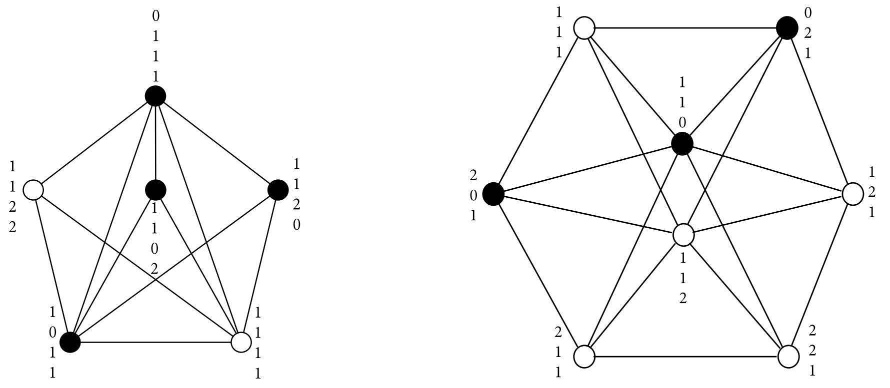

Fig. 6. A metric code of $W_{3,3}$ and $W_{2,6}$ with a metric basis.

Theorem 3.4. *For any $m, n \in Z^+$, every metric basis for the graph $W_{m,n}$ should include all of its central vertices except one.*

Lemma 3.5. *Let R be the set of rim vertices of the graph $G = W_{m,n}$. If S is a resolving set for G, then $|S \cap R| \geq \beta(W_{1,n})$ for all integers $n \neq 3, 6$ and $|S \cap R| = 2$ otherwise.*

Proof. Let $G = W_{m,n}$ and S be a minimal resolving set of G. We claim that $S \cap R \neq \emptyset$. Otherwise, $S \subseteq C$, where C is the set of all central vertices of G, so that S will not resolve any pair of rim vertices (since each rim vertex is adjacent to every central vertex), a contradiction to the assumption that S resolves G. Let $v_k \in C$ and $S' = (S - C) \cup \{v_k\}$. Now, as S is a resolving set of $W_{m,n}$, it follows that S' is a resolving set for the graph $W_{1,n}(= (G - V(C)) + v_k)$ containing a central vertex whenever $n \neq 3, 6$. However, since a metric basis of a wheel does not contain its central vertex (by Theorem 2.10), we conclude that $|S'| > \beta(W_{1,n})| \Rightarrow |S \cap R| = |S' - \{v_k\}| = |S'| - 1 \geq \beta(W_{1,n})$, for all $n \neq 3, 6$.

In the case when $n = 3, 6$, it is easy to observe that G has a resolving set S containing any two non-adjacent rim vertices and $m - 1$ central vertices for all $m \geq 2$ as in Fig. 6 shown below. Hence, $|S \cap R| = 2$ for all $m \geq 2$. □

Corollary 3.6. *Let C be the set of central vertices of the graph $G = W_{m,n}$, where $m \geq 2$ and $n \geq 3$. If S is a metric basis for G, then $|S| \geq \beta(W_{1,n}) + m - 1$ for $n \neq 3, 6$, and $|S| = m + 1$ for $n = 3, 6$.*

Proof. The result follows immediately by Theorem 3.4 and Lemma 3.5. □

Lemma 3.7. *Let C be the set of central vertices of the graph $G = W_{m,n}$, $m \geq 2$, $n \geq 3$. If S is a metric basis for G, then $|S| \leq \beta(W_{1,n}) + |C| - 1$. In particular, when $n = 3, 6$, $|S| \leq m + 1$.*

Proof. Let G' be the graph obtained from G by removing any $m-1$ of its m central vertices. Then $G' \cong W_{1,n}$. Let S' be a metric basis for G' and C and C' be the set of central vertices of G and G' respectively. Then, by Theorem 2.10, $S' \cap C = \emptyset$ (since the rim vertices of G

are the rim vertices of G'), $|C| = m$, $|C'| = 1$ and $|C \cap C'| = 1$. Define $S_1 = S' \cup (C - C')$. Due to distance hereditary property of vertices of G' in G, S' resolves all the pairs of vertices of G which are in G'. As for each of the remaining pairs of vertices of G, i.e., $u \in V(G)$ and a $v_k \in C - C'$, we have it resolved by v_k. Hence S_1 is a resolving set for G. Thus, for any metric basis S of G, we get $|S| \leq |S_1| = |S' \cup (C - C')| = |S'| + |C| - 1$ (since $S' \cap C = \emptyset$ and $|C'| = 1$) implies that $|S| \leq \beta(W_{1,n}) + |C| - 1$. Finally, in the case $n = 3, 6$, for $m \geq 2$, the set $S_2 = S_1 - \{v\}$ where v is any rim vertex of G is also resolving set for G. Therefore, for $n = 3, 6$, $|S| \leq |S_2| = |S_1| - 1 \leq \beta(W_{1,m}) + |C| - 1 - 1 = m + 1$. Hence the proof. □

From Lemma 3.7 and Corollary 3.6, we conclude the following;

Theorem 3.8. *For any two integers $m \geq 2, n \geq 3$,*

$$\beta(W_{m,n}) = \begin{cases} m + 1, & \text{if} \quad n = 3, 6 \\ \beta(W_{1,n}) + m - 1, & \text{if} \quad n \neq 3, 6 \end{cases}$$

To conclude, using Theorem 2.11 of B. Shanmukha et al. [7], the metric dimension of the generalized wheel is as follows.

Theorem 3.9. *For any positive integers $m \geq 1, n \geq 3$ and $n > m$,*

$$\beta(W_{m,n}) = \begin{cases} 3, & if \quad m = 1 \quad and \quad n = 3, 6 \\ m + 1, & if \quad m \geq 2 \quad and \quad n = 3, 6 \\ \lfloor \frac{5m+2n-3}{5} \rfloor, & if \quad m \geq 1 \quad and \quad n \neq 3, 6 \end{cases}$$

4. Metric dimension of corona product of graphs

In this section, we discuss some of the properties of the metric dimension of corona product of graphs. We also obtain the metric dimension of the corona product of any graph with some standard graphs.

Theorem 4.1 (*[1]*). $\beta(G) \leq \beta(G \odot K_1) \leq \beta(G) + 1$.

Observation 4.2. *If $|V(G)| = n$ and $|V(H)| = m$, then $|V(G \odot H)| = n(m + 1)$.*

Proof. Since $G \odot H$ contains n copies of H and a copy of G, so it contains $n \times (m + 1) = n(m + 1)$ vertices. □

Observation 4.3. *If G and H are any two graphs such that diameter of G is d, then $d + 2 \leq |V(G \odot H)| - \beta(G \odot H) \leq (d + 2)^{\beta(G \odot H)}$.*

Proof. If the diameter of G is d then we have the diameter of $G \odot H$ to be $d + 2$. The remaining part follows by Theorems 2.6 and 2.8. □

Lemma 4.4. *Let G and H be two connected graphs of order $n \geq 2$ and $m \geq 2$ respectively. Then for any resolving set S of $G \odot H$, $V(H_i) \cap S \neq \emptyset$, for every $i, 1 \leq i \leq n$.*

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of the graph G. Suppose to contrary that $G \odot H$ has a resolving set S with $V(H_i) \cap S = \emptyset$, for some $1 \leq i \leq n$. Since $m \geq 2$, for any two vertices $x, y \in V(H_i)$ and for every vertex $u \in S$, we have

$$\begin{aligned} d(x, u) &= d(x, v_i) + d(v_i, u) \\ &= d(y, v_i) + d(v_i, u) \\ &= d(y, u), \end{aligned}$$

a contradiction to the fact that S is a resolving set. □

Lemma 4.5. *Let G and H be two connected graphs of order $n \geq 2$ and $m \geq 2$ respectively. Then for any metric basis S of $G \odot H$, $V(G) \cap S = \emptyset$.*

Proof. Let G and H be connected graphs and S be a metric basis for $G \odot H$. Let $M = S - V(G)$. In order to prove that $M = S$, we show that M is a resolving set for $G \odot H$.

Let x and y be any two vertices of $G \odot H$. Then, we have the following cases;

Case 1: $x, y \in V(H_i)$

As every vertex $u \in G \odot H$ which are not in $V(H_i)$ is equidistant from x and y, there exists a vertex $v \in V(H_i) \cap M$ such that $d(x, v) \neq d(y, v)$.

Case 2: $x, y \in V(G)$

Let $x = v_i$ and $y = v_j$. In this case, by Lemma 4.4, we have $v \in V(H_i) \cap M$. Further, $d(x, v) = d(v_i, v) = 1 < 1 + d(v_j, v_i) = d(y, v)$ so that $d(x, v) \neq d(y, v)$.

Case 3: $x \in V(H_i)$ and $y \in V(H_j)$, for $i \neq j$

Again, by Lemma 4.4, we have $v \in V(H_i) \cap M$. For this vertex v, $d(x, v) \leq 2 < 3 \leq d(y, v)$ so that $d(x, v) \neq d(y, v)$.

Case 4: $x \in V(H_i)$ and $y \in V(G)$.

Subcase 1: x is adjacent to y.

In this case, $y = v_i$. Choose an existing vertex $v \in V(H_j) \cap M$. For this vertex, $d(x, v) = 1 + d(y, v) > d(y, v)$. Hence $d(x, v) \neq d(y, v)$.

Subcase 2: x is not adjacent to y

Let $y = v_k$ where $k \neq i$. Choose an existing vertex $v \in V(H_k) \cap M$. For this vertex $d(x, v) = d(x, y) + d(y, v) > d(y, v)$. Hence $d(x, v) \neq d(y, v)$.

Thus, M is a resolving set for $G \odot H$ so that S is not a metric basis (being a minimal resolving set with minimum cardinality) of $G \odot H$ unless $M = S$. Therefore, $V(G) \cap S = \emptyset$. □

Lemma 4.6. *Let G and H be two connected graphs and S be a resolving set for $G \odot H$. Then for $1 \leq i \leq n$, $S \cap V(H_i)$ is a resolving set of H_i.*

Proof. Let $S_i = S \cap V(H_i)$. Then, by Lemma 4.4, $S_i \neq \emptyset$. Let us suppose that $x, y \in V(H_i) - S_i$. Then, as S is a resolving set of $G \odot H$, there exists $u \in S$ which resolves x and y (i.e. $r(x|S) \neq r(y|S)$). But for every vertex $u \in S - V(H_i)$, we have

$$\begin{aligned} d(x,u) &= d(x,v_i) + d(v_i,u) \\ &= d(y,v_i) + d(v_i,u) \\ &= d(y,u). \end{aligned}$$

Therefore, to resolve $x, y \in V(H_i)$, the resolving vertex $u \in S \cap V(H_i) = S_i$. Thus, S_i is a resolving set for H_i. □

Remark 4.7. The vertex u in the proof of Lemma 4.6 is adjacent to exactly one of the two vertices x and y since u resolves x and y. In particular, if u is adjacent to x and not adjacent to y, then $d_{G\odot H}(x,u) = d_{H_i}(x,u) = 1$ and $d_{G\odot H}(y,u) = 2 \leq d_{H_i}(y,u)$. The other case follows similarly.

Theorem 4.8. *If G and H are two graphs of order n and m respectively with $m, n \geq 2$, then $\beta(G \odot H) \geq n\beta(H)$. In particular, if $diam(H) = 2$, $\beta(G \odot H) = n\beta(H)$.*

Proof. Let G be a graph and S be a metric basis for $G \odot H$. Let $\{v_1, v_2, \ldots, v_n\}$ be the set of vertices of G and H_i be the copy of H at the vertex v_i of G for each i, $1 \leq i \leq n$. Then by Lemma 4.5, we have $S \cap V(G) = \phi$ and by Lemma 4.6, $S \cap V(H_i) = S_i$ is a resolving set for $V(H_i)$, $i = 1, 2, 3, \ldots, n$. Therefore, $\bigcup_{i=1}^{n} S_i$ is a resolving set for $G \odot H$. Hence $|S| = |\bigcup_{i=1}^{n} S_i| = \sum_{i=1}^{n} |S_i| = n|S_i| \geq n\beta(H_i) = n\beta(H) \Rightarrow \beta(G \odot H) \geq n\beta(H)$.

Further, if $diam(H) = 2$, then $d_H(u,v) = d_{G\odot H}(u,v)$ for all $u, v \in V(H)$. Let S' be a metric basis for H, S_i be the copy of the set S' of vertices in H_i. Consider the set $M = \bigcup_{i=1}^{n} S_i$. We show that S will resolve all the vertices in $G \odot H$. Let $u, v \in V(G \odot H)$ be arbitrary. Then we have the following cases;

Case 1: $u, v \in V(H_i)$, for some i, $1 \leq i \leq n$.
In this case $w \in S_i$ will resolve u and v (by the choice of S_i and $d_H(u,v) = d_{G\odot H}(u,v)$ for all $u, v \in H$).

Case 2: $u, v \in V(G)$.
In this case, as S resolves G, there is a vertex $w \in S$ which resolves u and v. Choose a vertex $u' \in S_k \cap M$, where H_k is the copy of H at a vertex w. For this vertex u', we have

$$\begin{aligned} d(u',u) &= d(u',w) + d(w,u) \\ &= 1 + d(w,u) \\ &\neq 1 + d(w,v) \\ &= d(u',w) + d(w,v) \\ &= d(u',v) \end{aligned}$$

Hence there exists $u' \in M$ which resolves u and v.

Case 3: $u \in V(G)$ and $v \in V(H_j)$, for some j, $1 \leq j \leq n$.
In this case, $u = v_i$, for some i, $1 \leq i \leq n$.

Subcase 1: $i = j$.
In this case, choose an existing vertex $w \in V(H_k) \cap M$, where $k \neq j$. For this vertex, $d(v,w) = 1 + d(v,w) > d(v,w)$. Hence $d(u,w) \neq d(v,w)$, w will resolve u and v in this case

Subcase 2: $i \neq j$.
Similar to the above subcase, the existing vertex $w \in V(H_i) \cap M$ will resolve u and v in this case.

Hence in each of the above cases, there exists $w \in M$ which resolves u and v, for each pair $u, v \in G \odot H$ whenever $diam(H) = 2$ so that it is a resolving set for $G \odot H$. Thus, $|M| \geq \beta(G \odot H) \Rightarrow |\bigcup_{i=1}^{n} S_i| \geq \beta(G \odot H) \Rightarrow \sum_{i=1}^{n} |S_i| \geq \beta(G \odot H) \Rightarrow \sum_{i=1}^{n} |S'| \geq \beta(G \odot H) \Rightarrow \sum_{i=1}^{n} \beta(H) \geq \beta(G \odot H) \Rightarrow n\beta(H) \geq \beta(G \odot H)$.

Thus, as $n\beta(H) \leq \beta(G \odot H)$ and $n\beta(H) \geq \beta(G \odot H)$, it follows that $n\beta(H) = \beta(G \odot H)$ whenever $diam(H) = 2$. $\square$

Using the results on the metric dimension of some known graphs, we conclude the following;

Corollary 4.9. *For any non-trivial connected graph G of order n*

1. $\beta(G \odot P_m) = n$, *whenever* $2 \leq m \leq 3$.
2. $\beta(G \odot C_m) = 2n$, *whenever* $3 \leq m \leq 5$.
3. $\beta(G \odot K_m) = n(m-1)$, *whenever* $m \geq 2$.
4. $\beta(G \odot K_{r,s}) = n(r+s-2)$.
5. $\beta(G \odot W_{1,m}) = \begin{cases} 3n, & if \quad m = 3, 6 \\ 2n, & if \quad m = 4, 5 \\ \lfloor \frac{2m+2}{5} \rfloor n, & if \quad m \geq 7 \end{cases}$.
6. *For the fan graph $F_{1,m}$, $m \geq 4$,*

$$\beta(G \odot F_{1,m}) = \begin{cases} 3n, & if \quad m = 6 \\ \lfloor \frac{2m+2}{5} \rfloor n, & \quad otherwise \end{cases}.$$

Theorem 4.10. *Let G be a connected graph of order $n \geq 2$ and let H be a graph of order $m \geq 2$. Then if $n\beta(H) \leq \beta(G \odot H) \leq n\beta(K_1 \odot H)$, $\beta(H) \leq \beta(K_1 \odot H)$.*

Proof. The lower bound follows by Theorem 4.8. To prove the upper bound, we first see that $K_1 \odot H_i$ is the subgraph of $G \odot H$ obtained by joining the vertex $v_i \in V(G)$ with all vertices of H_i. Now, for every $v_i \in V(G)$, let S_i be a metric basis of $K_1 \odot H_i$ and let $S = \cup_{i=1}^{n} S_i$. By Lemma 4.5, v_i does not belong to any basis for $K_1 \odot H_i$. So S does not contain any vertex from G. We show that S is a resolving set for $G \odot H$. Let x, y be two distinct vertices of $G \odot H$.

Case 1: $x, y \in V(H_i)$.
In this case there exists $u \in S_i$ such that $d_{K_1 \odot H_i}(x, v) \neq d_{K_1 \odot H_i}(y, u)$ which shows $d_{G \odot H}(x, v) \neq d_{G \odot H}(y, u)$.

Case 2: $x \in V(H_i)$ and $y \in V(H_j)$ and $i \neq j$.
In this case, let $v \in S_i$, then we have $d_{G \odot H}(x, v) \leq 2 < 3 \leq d_{G \odot H}(y, v)$

Case 3: $x, y \in V(G)$.
In this case, let x be adjacent to the vertices of H_i which is a copy at x. So, for every $v \in S_i$, we have $d_{G \odot H}(x, v) = 1 < d_{G \odot H}(y, x) + 1 = d_{G \odot H}(y, v)$.

Case 4: $x \in V(H_i)$ and $y \in V(G)$.
If x is adjacent y, then for every $v \in S_j$, $j \neq i$, we have $d_{G \odot H}(x, v) = 1 + d_{G \odot H}(y, v) > d_{G \odot H}(y, v)$. Else x is not adjacent to y, so there exists $v \in S_j$

adjacent to y and $j \neq i$, we have $d_{G \odot H}(x, v) = d_{G \odot H}(x, y) + 1 = d_{G \odot H}(x, y) + d_{G \odot H}(y, v) > d_{G \odot H}(y, v)$.

Then there exists $x, y \in G \odot H$ such that $r(x|S) \neq r(y|S)$. It gives $\beta(G \odot H) \leq n\beta(K_1 \odot H)$. Hence we have $n\beta(H) \leq \beta(G \odot H) \leq n\beta(K_1 \odot H)$. □

5. A GENERALIZATION OF CORONA PRODUCT OF GRAPHS

In this section, we obtain bounds on the metric dimension of generalized corona product of graphs.

Lemma 5.1. *Let G be a connected graph of order n and H be any graph having n ordered components $H_1, H_2, \ldots, H_n$. Let S be a metric basis of $G \odot' H$. Then there exists $M \subseteq V(G \cap H)$ which resolves vertices in G such that $|M| \leq |S|$.*

Proof. Let $M = \bigcup_{i=1}^{n} \{u_i \in V(G) : u_i$ be a maximum degree vertex in H_i and $S \cap H_i \neq \emptyset\}$. We show that M is a resolving set for G.

Let x, y be any two vertices of G. Since G is a subgraph of $G \odot' H$, the vertices x, y are vertices of $G \odot' H$ and hence there exists w in the metric basis S of G such that $d(x, w) \neq d(y, w)$. But then, $w \in V(H_i)$, for some $i, 1 \leq i \leq n$. Suppose u_i be the vertex of H_i common to G in $G \odot' H$, then $d(x, u_i) + d(u_i, w) = d(x, w) \neq d(y, w) = d(y, u_i) + d(u_i, w)$ so that $d(x, u_i) \neq d(y, u_i)$ and $u_i \in M$. (Because each u_i is a cut vertex and H_i is a block of the graph $G \odot' H$ containing u_i). Hence $u_i \in M$ resolves x and y. Since x and y are the arbitrary vertices in G, we conclude that M is a resolving set for G. Finally, by the construction of M and the fact that $S \cap H_i \neq \emptyset$, if $m \in M$, then there exists $m' \in S \cap H_i$ so that $m' \in S$. Further, this m' will not correspond to any other $m \in M$. Hence $|M| \leq |S|$. □

Lemma 5.2. *Let S be a metric basis for the graph G of order n and S' be any subset of $V(G \odot' H)$ such that $|S'| < |S|$. Then there exists a pair of vertices $u, v \in V(G)$ with the property that $d(u, w) = d(v, w)$ for every $w \in S'$.*

Proof. On the contrary, suppose that for every pair of vertices $u, v \in V(G \odot' H) - S'$, we can find a $w \in S'$ such that $d(u, w) \neq d(v, w)$. Then, S' is resolving set of $G \odot' H$. However, by above Lemma 5.1 we get a resolving set M for G with $|M| \leq |S'| < |S|$ which is a contradiction to the fact that S is a metric basis of G. □

Theorem 5.3. *Let G be a connected graph of order n and $H_1, H_2, \ldots, H_n$ be the components of a graph H. Then we have*

$$\beta(G) \leq \beta(G \odot' H) \leq \sum_{i=1}^{n} \beta(H_i).$$

Proof. If G and H are trivial graphs, then $G \odot' H$ is a trivial graph and hence $\beta(G \odot' H) = 1$.

Otherwise, let G be a graph of order n and H_i be the ith component of H with u_i being a maximum degree vertex in H_i. Then we have the following cases.

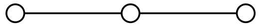

Fig. 7. The graph G.

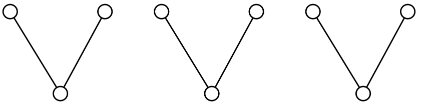

Fig. 8. The graph H.

Case 1: H is a totally disconnected graph.
In this case, $G \odot' H \cong G$ and hence, $\beta(G \odot' H) = \beta(G) \leq n-1 < n = \sum_{i=1}^{n} 1 = \sum_{i=1}^{m} \beta(H_i)$.

Case 2: $H_i \cong K_2$ for each i, $1 \leq i \leq n$.
In this case $G \odot' H \cong G \odot K_1$ and hence by Theorem 4.1, its follows that $\beta(G) \leq \beta(G \odot' H) \leq \beta(G) + 1 \leq n - 1 + 1 = n = \sum_{i=1}^{n} 1 = \sum_{i=1}^{m} \beta(H_i)$.

Case 3: $H_i \ncong K_1, K_2$ for some i, $i \leq i \leq n$.
Let S be a metric basis of G and S' be any subset of vertices of $G \odot' H$ such that $|S'| \leq |S|$. Then by Lemma 5.2, S' is not a resolving set of $G \odot' H$. Therefore, $\beta(G \odot' H) > |S'|$. Hence $\beta(G \odot' H) \geq |S| = \beta(G)$.

Thus, we see that $\beta(G) \leq \beta(G \odot' H)$ for all the graphs G.

Now to prove the other inequality for the cases $H_i \ncong K_1, K_2$ for any i, $1 \leq i \leq n$, let S_i be the metric basis of H_i, the ith component of H and $S = \bigcup_{i=1}^{n} S_i$.

Claim : S is a resolving set of $\beta(G \odot' H)$.
Let u and v be any two vertices of $G \odot' H$.

Subcase (i): Both $u, v \in V(H_i)$ for some i, $1 \leq i \leq n$. Since S_i is metric basis for the component H_i of H and $u, v \in V(H_i)$, we can find $w \in S \cap V(H_i)$ such that $d(u, w) \neq d(v, w)$.

Subcase (ii): $u \in V(H_i)$ and $v \in V(H_j)$ for some i, j $(i \neq j)$. If $v \neq v_j$, then for $w \in S \cap V(H_i)$, $d(u, w) \leq 2$ and $d(v, w) > 2$ so that $d(u, w) \neq d(v, w)$. If $v = v_j$, then for $w \in S \cap V(H_j)$, $d(u, w) \geq 2$ and $d(v, w) = 1$ and hence $d(u, w) \neq d(v, w)$.

Thus for any two vertices u and v of $G \odot' H$, we have $w \in S$ such that $d(u, w) \neq d(v, w)$.

Since $S = \cup_{i=1}^{n} S_i$, S_i is the metric basis of H_i and by the above claim, it follows that, we have $\beta(G \odot' H) \leq |S| = |\sum_{i=1}^{n} S_i| = \sum_{i=1}^{n} |S_i| = \sum_{i=1}^{n} \beta(H_i)$. $\square$

Remark 5.4. Let $G = P_3$ and H be the graph having components $H_i = P_3$, $1 \leq i \leq 3$. Then $\beta(G) = 3$ and $\beta(G \odot' H) = 3$ (Fig. 9). This shows that the bound obtained for $\beta(G \odot' H)$ in the above Theorem 5.3 is tight.

6. Conclusion

The metric dimension of the family of generalized wheels has been obtained in this paper. Also, some results on the metric bases of the corona product of graphs have been established

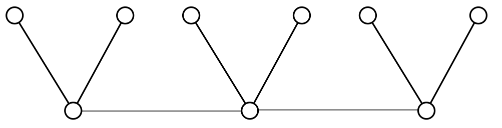

Fig. 9. The graph $G \odot' H$ of the graphs G of Fig. 7 and H of Fig. 8.

using which the metric dimension of the corona product of any graph G with some standard graphs has been obtained. Further, some relations between the metric dimension of a graph and its generalized corona product have been established.

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