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The solution of certain triple q-integral equations in fractional q-calculus approach

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Abstract. In this paper, we consider a certain system of triple q-integral equations, where the kernel is the third Jackson q-Bessel functions. We give two solutions by using the fractional q-calculus approach. We study also the system with general kernel. A q-analogue of the result by Cooke and by Williams of 1963 is included.

Keywords: q-special functions; Fractional q-integral operator; Triple q-integral equations

Mathematics Subject Classification: primary 45F10; secondary 31B10; 26A33; 33D45

1. Introduction

Multiple integral equations have many applications, where some mixed boundary value problems of the mathematical theory of elasticity are solved by reducing them to dual and triple integral equations. For example, harmonic shear oscillations of a rigid stamp with a plane base coupled to an elastic half-space were studied in [26] and reduced to dual integral equations. In [14], the authors determined the stress distribution in a two-dimensional medium containing a circular hole and two symmetrically situated Griffith cracks by using triple integral equations.

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So in the past years, several authors have solved certain systems of dual and triple integral equations (especially when the kernel is a Bessel function) in different ways where some results and a number of problems were discussed in well-known papers. For the dual integral equations, see [9,11,15,18–21], and for the triple integral equations, see [5–8,10,16,24].

In [7], Cooke solved the following triple integral equations by operational methods.

$$\int_0^\infty \phi(\xi)J_\nu(\xi x)\,d\xi = G_1(x), \qquad 0 < x < a, \tag{1.1}$$

$$\int_0^\infty \xi^{-2\alpha}\phi(\xi)J_\nu(\xi x)\,d\xi = F_2(x), \qquad a < x < b, \tag{1.2}$$

$$\int_0^\infty \phi(\xi)J_\nu(\xi x)\,d\xi = G_3(x), \qquad b < x < \infty, \tag{1.3}$$

in which G_1, F_2 and G_3 are known functions and ϕ is an unknown function to be determined. This system generalized [6] and [25] when G_1 and G_3 are non zero functions. He also solved Eqs. (1.1)–(1.3) with the extra factor $1 + k(\xi)$ in the kernel of (1.2), where k is a known function.

In this paper, we give a q-type analogue of Cooke's problem. More precisely, the principle result is given by

Theorem 1.1. *Let α and μ be complex numbers such that $\Re(\mu) > -1$ and $\Re(\mu - \alpha) > -1$. Assume that $g_1 \in L_{q^2,\frac{\mu}{2}+\alpha}(A_{q^2,a})$, $f_2 \in L_{q^2,\frac{\mu}{2}+\alpha}(A_{q^2,b}) \cap L_{q^2,-\frac{\mu}{2}+\alpha-1}(B_{q^2,a})$ and $g_3 \in L_{q^2,-\frac{\mu}{2}+\alpha-1}(B_{q^2,b})$ are known functions, where $0 < a < b < \infty$. Then the system of the triple q^2-integral equations:*

$$P\psi \equiv \int_0^\infty \psi(\rho)J_\mu(\sqrt{\rho\xi};q^2)\,d_{q^2}\rho = g_1(\xi), \quad \xi \in A_{q^2,a}, \tag{1.4}$$

$$Q\psi \equiv \xi^{-\alpha}\int_0^\infty \rho^{-\alpha}\psi(\rho)J_\mu(\sqrt{\rho\xi};q^2)\,d_{q^2}\rho = f_2(\xi), \quad \xi \in A_{q^2,b} \cap B_{q^2,a}, \tag{1.5}$$

$$P\psi \equiv \int_0^\infty \psi(\rho)J_\mu(\sqrt{\rho\xi};q^2)\,d_{q^2}\rho = g_3(\rho), \quad \xi \in B_{q^2,b}, \tag{1.6}$$

is reduced to a pair of q-simultaneous equations, where $\psi \in L_{q^2,\frac{\mu}{2}-\alpha}(\mathbb{R}_{q^2,+}) \cap L_{q^2,\mu}(\mathbb{R}_{q^2,+})$ is to be determined.

It is worth mentioning that different approaches for solving dual and triple q-integral equations are in [4] and [17].

Throughout this paper, we assume that q is a positive number less than one. For $t > 0$, the sets $A_{q,t}$, $B_{q,t}$ and $\mathbb{R}_{q,t,+}$ are defined by

$$A_{q,t} := \{tq^n : n \in \mathbb{N}_0\}, \quad B_{q,t} := \{tq^{-n} : n \in \mathbb{N}\},$$

$$\mathbb{R}_{q,t,+} := \{tq^k : k \in \mathbb{Z}\},$$

where $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$, and $\mathbb{N} := \{1, 2, \ldots\}$. Notice, if $t = 1$ we write A_q, B_q, and $\mathbb{R}_{q,+}$ and we define the following spaces:

$$L_{q,\eta}(\mathbb{R}_{q,+}) := \Big\{f : \|f\|_{q,\eta} := \int_0^\infty |t^\eta f(t)|d_qt < \infty\Big\},$$

$$L_{q,\eta}(A_q) := \Big\{f : \|f\|_{A_q,\eta} := \int_0^1 |t^\eta f(t)|d_qt < \infty\Big\},$$

$$L_{q,\eta}(B_q) := \left\{ f : \|f\|_{B_q,\eta} := \int_1^{\infty} |t^{\eta} f(t)| d_q t < \infty \right\},$$

where $\eta \in \mathbb{C}$ and $L_{q,\eta}(\mathbb{R}_{q,+}) = L_{q,\eta}(A_q) \cap L_{q,\eta}(B_q)$.

We follow Gasper and Rahman [12] for the definitions of Jackson q-integrals and we recall some notions and definitions of the q-calculus (see, *e.g.* [3]).

The q-shifted fractional is defined by

$$(a; q)_{\infty} = \prod_{j=0}^{\infty} (1 - aq^j), \quad \text{and} \quad (a; q)_n := \frac{(a; q)_{\infty}}{(aq^n; q)_{\infty}} \quad \text{for} \quad n \in \mathbb{Z},\ a \in \mathbb{C}.$$

The q-gamma function is defined by

$$\Gamma_q(z) = \frac{(q; q)_{\infty}}{(q^z; q)_{\infty}} (1 - q)^{1-z} \quad \text{for} \quad z \in \mathbb{C} \setminus \{-n : n \in \mathbb{N}_0\}.$$

The q-derivative $D_q f$ of an arbitrary function f is given by

$$(D_q f)(x) := \frac{f(x) - f(qx)}{(1 - q)x}, \quad x \neq 0.$$

In particular, if $f(x) = (x/t; q)_{\alpha}$, then

$$(D_q f)(x) = -\frac{[\alpha]}{t} (qx/t; q)_{\alpha-1}, \quad \text{where } [\alpha] = \frac{1 - q^{\alpha}}{1 - q}. \tag{1.7}$$

The third Jackson q-Bessel function $J_{\nu}^{(3)}(z; q)$ is defined by

$$J_{\nu}(z; q) = J_{\nu}^{(3)}(z; q) := \frac{(q^{\nu+1}; q)_{\infty}}{(q; q)_{\infty}} \sum_{n=0}^{\infty} (-1)^n \frac{q^{n(n+1)/2} z^{2n+\nu}}{(q; q)_n (q^{\nu+1}; q)_n}, \ z \in \mathbb{C}.$$

The q-integration by parts rule on an interval $[a, b]$ is

$$\int_a^b f(t)(D_q g)(t)\, d_q t = [(fg)(t)]_a^b - \int_a^b g(qt)(D_q f)(t)\, d_q t.$$

Koornwinder and Swarttouw [13] introduced the following inverse pair of q-Hankel integral transforms under the side condition $f,\ g \in L_q^2(\mathbb{R}_{q,+})$:

$$g(\lambda) = \int_0^{\infty} x f(x) J_{\nu}(\lambda x; q^2)\, d_q x; \quad f(x) = \int_0^{\infty} \lambda g(\lambda) J_{\nu}(\lambda x; q^2)\, d_q \lambda, \tag{1.8}$$

where $\lambda,\ x \in \mathbb{R}_{q,+}$.

This paper is organized as follows. In Section 2 we give some background on fractional q-integral operators and their properties which we need in our fractional q-calculus approach for solving the system in Theorem 1.1. In Section 3, we give two solutions of Eqs. (1.4)–(1.6) by reducing the system to two simultaneous Fredholm q-integral equations of the second kind, we shall use a method due to [7]. In particular, we introduce q-analogue of the results introduced by Cooke in [5] and Williams [25]. In Section 4, we consider the case in which the integrand in (1.5) has an additional factor $1 + w(\rho)$, where w is a known function.

2. Preliminaries on fractional q-calculus

In this section, we recall some of fractional q-integral operators and their properties (see [1,2,4] and [22]).

A q-analogue of the Riemann–Liouville fractional integral operator is introduced in [2] by Al-Salam through

$$I_q^{\alpha} f(x) := \frac{x^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^x (qt/x;q)_{\alpha-1} f(t)\, d_qt, \quad \alpha \notin \{-1,-2,\ldots\}.$$

In [1], Agarwal defined the two parameter family fractional q-integral operator

$$I_q^{\nu,\alpha}\phi(x) := \frac{x^{-\nu-1}}{\Gamma_q(\alpha)} \int_0^x (qt/x;q)_{\alpha-1}\, t^{\nu}\phi(t)\, d_qt, \quad \alpha \notin \{-1,-2,\ldots\}. \tag{2.1}$$

This operator can be written as

$$I_q^{\nu,\alpha}\phi(x) = (1-q)^{\alpha} \sum_{n=0}^{\infty} q^{(\nu+1)n} \frac{(q^{\alpha};q)_n}{(q;q)_n}\, \phi(xq^n), \tag{2.2}$$

which is valid for all α.

Also, Al-Salam (see [2]) defined a two parameter q-fractional operator by

$$K_q^{\eta,\alpha}\phi(x) := \frac{q^{-\eta}x^{\eta}}{\Gamma_q(\alpha)} \int_x^{\infty} \big(x/t;q\big)_{\alpha-1} t^{-\eta-1}\phi(tq^{1-\alpha})\, d_qt, \quad \alpha \notin \{-1,-2,\ldots\}.$$

This is a q-analogue of the Erdélyi and Sneddon fractional operator.

In [4], the authors introduced a slight modification of the operator $K_q^{\eta,\alpha}$. This operator is denoted by $\mathcal{K}_q^{\eta,\alpha}$ and defined by

$$\mathcal{K}_q^{\eta,\alpha}\phi(x) := \frac{q^{-\eta}x^{\eta}}{\Gamma_q(\alpha)} \int_x^{\infty} (x/t;q)_{\alpha-1} t^{-\eta-1}\phi(qt) d_qt, \quad \alpha \notin \{-1,-2,\ldots\}. \tag{2.3}$$

It satisfies the semigroup property

$$\mathcal{K}_q^{\eta,\alpha}\mathcal{K}_q^{\eta+\alpha,\beta}\phi(x) = \mathcal{K}_q^{\eta,\alpha+\beta}\phi(x), \tag{2.4}$$

where $x \in \mathbb{R}_{q,+}$ whenever $\phi \in L_{q,-\eta-\alpha-1}(B_q)$, $\Re(\alpha) < 0$ and η, α, β are complex numbers.

We denote by $I_{q,c,d}^{\eta,\alpha}\, \psi(x)$ and by $\mathcal{K}_{q,e,f}^{\eta,\alpha}\, \phi(x)$ to the following operators:

$$I_{q,c,d}^{\eta,\alpha}\, \psi(x) := \frac{x^{-\eta-1}}{\Gamma_q(\alpha)} \int_c^d (qt/x;q)_{\alpha-1}\, t^{\eta}\psi(t)\, d_qt, \quad 0 < c < d < x$$

$$\mathcal{K}_{q,e,f}^{\eta,\alpha}\, \phi(x) := \frac{q^{-\eta}x^{\eta}}{\Gamma_q(\alpha)} \int_e^f (x/t;q)_{\alpha-1} t^{-\eta-1}\phi(qt) d_qt, \quad x < e < f.$$

Note that $I_{q,c}^{\eta,\alpha}$ generalized the classical definition of $I_q^{\eta,\alpha}$; by shifting the point of the origin from zero to be a constant $c \in \mathbb{R}$. Also, for the operator $\mathcal{K}_{q,d}^{\eta,\alpha}$, the upper limit of the integration of $\mathcal{K}_q^{\eta,\alpha}$ is replaced by some constant $d \in \mathbb{R}$.

We have the following properties (its proof can be found in [4] and [22]).

$$(I_{q,c}^{\eta,\alpha})^{-1}\psi = I_{q,c}^{\eta+\alpha,-\alpha}\psi, \quad 0 \le c < x, \tag{2.5}$$

$$(\mathcal{K}_{q,d}^{\eta,\alpha})^{-1}\phi = \mathcal{K}_{q,d}^{\eta+\alpha,-\alpha}\phi, \quad d > x, \tag{2.6}$$

for any $\eta, \alpha \in \mathbb{C}$, $\psi \in L_{q,\eta+\alpha}(A_q)$ and $\phi \in L_{q,-\eta-\alpha-1}(B_q)$.

By using (1.7) and the q-integration by parts, it is easy to verify that

$$I_{q,c}^{\eta,\alpha}\phi(x) = x^{-\eta-\alpha}\, I_{q,c}^{\alpha+1} D_q[(\cdot)^{\eta}\phi(\cdot)](x) + \frac{c^{\eta}\, x^{-\eta}}{\Gamma_q(\alpha+1)}(c/x;q)_{\alpha}\, \phi(c). \tag{2.7}$$

Now, we will define two operators $L_q^{\eta,\alpha}$ and $M_q^{\eta,\alpha}$ by the following equations, which we found very convenient to use in our analysis:

$$
\begin{aligned}
{}_cL_{q,d,e}^{\eta,\alpha}\,\psi(x) &:= (I_{q,c}^{\eta,\alpha})^{-1}\,I_{q,d,e}^{\eta,\alpha}\,\psi(x), \qquad x\in A_q,\ d<e\le c.\\
{}_fM_{q,g,h}^{\eta,\alpha}\,\phi(x) &:= (\mathcal{K}_{q,f}^{\eta,\alpha})^{-1}\,\mathcal{K}_{q,g,h}^{\eta,\alpha}\phi(x), \qquad x\in B_q,\ f<g\le h.
\end{aligned}
$$

Lemma 2.1. *Let η,α be complex numbers and $-1<\Re(\alpha)<1$, $\psi\in L_{q,\eta+\alpha}(A_q)$. Then*

$$
\begin{aligned}
{}_cL_{q,d,e}^{\eta,\alpha}\,\psi(x) = {}&\frac{[\alpha-1]x^{-\eta-\alpha}}{\Gamma_q(\alpha)\Gamma_q(1-\alpha)}\int_d^e y^\eta K_\alpha(y,t)\psi(y)\,d_qy\\
&+\frac{c^{\eta+\alpha}\,x^{-\eta-\alpha}}{\Gamma_q(1-\alpha)}(c/x;q)_{-\alpha}\,I_{q,d,e}^{\eta,\alpha}\psi(c),
\end{aligned}
$$

where

$$
K_\alpha(y,t)=\int_c^x t^{\alpha-2}\,(qt/x;q)_{-\alpha}(qy/t;q)_{\alpha-2}\,d_qt.
$$

Proof. Using (2.5) and formula (2.7), it follows that

$$
\begin{aligned}
&{}_cL_{q,d,e}^{\eta,\alpha}\,\psi(x)\\
&:=\frac{x^{-\eta-\alpha}}{\Gamma_q(\alpha)\Gamma_q(1-\alpha)}\int_c^x (qt/x;q)_{-\alpha}D_q\int_d^e (qy/t;q)_{\alpha-1}\,t^{\alpha-1}y^\eta\psi(y)\,d_qy\,d_qt\\
&\quad+\frac{c^{\eta+\alpha}\,x^{-\eta-\alpha}}{\Gamma_q(1-\alpha)}(c/x;q)_{-\alpha}\,I_{q,d,e}^{\eta,\alpha}\psi(c)\\
&=\frac{[\alpha-1]x^{-\eta-\alpha}}{\Gamma_q(\alpha)\Gamma_q(1-\alpha)}\int_c^x (qt/x;q)_{-\alpha}\int_d^e (qy/t;q)_{\alpha-2}\,t^{\alpha-2}y^\eta\psi(y)\,d_qy\,d_qt\\
&\quad+\frac{c^{\eta+\alpha}\,x^{-\eta-\alpha}}{\Gamma_q(1-\alpha)}(c/x;q)_{-\alpha}\,I_{q,d,e}^{\eta,\alpha}\psi(c).
\end{aligned}
$$

Interchanging the order of the double q-integration, which is allowable by the absolute convergence of q-integrals, we get the desired result. □

Similarly, we have the following:

Lemma 2.2. *Let η,α be complex numbers and $-1<\Re(\alpha)<1$, $\phi\in L_{q,\eta+\alpha}(B_q)$. Then*

$$
\begin{aligned}
{}_fM_{q,g,h}^{\eta,\alpha}\,\phi(x) = {}&-\frac{[\alpha-1]q^{1-\eta-\alpha}x^{\eta+\alpha}}{\Gamma_q(\alpha)\Gamma_q(1-\alpha)}\int_g^h y^{-\eta-2}\hat{K}_\alpha(y,t)\phi(y)\,d_qy\\
&+\frac{(qf)^{-\eta-\alpha}\,x^{\eta+\alpha}}{\Gamma_q(1-\alpha)}(x/f;q)_{-\alpha}\,\mathcal{K}_{q,g,h}^{\eta,\alpha}\phi(qf),
\end{aligned}
$$

where

$$
\hat{K}_\alpha(y,t)=\int_x^f (qt)^{-\alpha}\,(x/qt;q)_{-\alpha}(q^2t/y;q)_{\alpha-2}\,d_qt.
$$

Furthermore, we can get the following equations (see the Appendix in [7]):

$$
I_{q,c,d}^{\eta+\alpha,-\alpha}\,I_{q,c}^{\eta,\alpha}\psi(x)=-\,{}_dL_{q,c,d}^{\eta,\alpha}\,\psi(x), \qquad x>d>c. \tag{2.8}
$$

$$
\mathcal{K}_{q,f,g}^{\eta+\alpha,-\alpha}\,\mathcal{K}_{q,g}^{\eta,\alpha}\phi(x)=-\,{}_fM_{q,f,g}^{\eta,\alpha}\,\phi(x), \qquad x<g<f. \tag{2.9}
$$

We end this section with the definition and some results of a q-analogue of the modified Hankel transform operator introduced by Erdélyi and Kober (for more details, see [4]).

Let $S_q^{\eta,\alpha}$ be the operator defined by

$$S_q^{\eta,\alpha}\phi(x) := \frac{x^{-\alpha/2}}{(1-q)}\int_0^\infty y^{-\alpha/2}\phi(y)J_{2\eta+\alpha}(\sqrt{xy};q)\,d_q y$$
$$= x^{-\alpha/2}\sum_{n=-\infty}^{\infty} q^{n(1-\alpha/2)}J_{2\eta+\alpha}(\sqrt{x}q^{n/2};q)\phi(q^n). \tag{2.10}$$

Proposition 2.3. *Let η and α be complex numbers, and $\Re(2\eta+\alpha) > -1$. If $\phi \in L_{q^2,\eta}(\mathbb{R}_{q^2,+})$, then $S_{q^2}^{\eta,\alpha}\phi(x)$ exists for all $x \in \mathbb{R}_{q^2,+}$ and belong to $L_{q^2,\eta+\alpha}(\mathbb{R}_{q^2,+})$.*

Proposition 2.4. *$S_{q^2}^{\eta,\alpha}$ defines a one to one linear operator from $L_{q^2,\eta}(\mathbb{R}_{q^2,+})$ into $L_{q^2,\eta+\alpha}(\mathbb{R}_{q^2,+})$. Also,*

$$(S_q^{\eta,\alpha})^{-1} = S_q^{\eta+\alpha,-\alpha} \tag{2.11}$$

Proposition 2.5. *Let α, β and η be complex numbers such that $\Re(2\eta+\alpha) > -1$. If $\phi \in L_{q^2,\eta}(\mathbb{R}_{q^2,+})$, then*

$$I_{q^2}^{\eta+\alpha,\beta}\, S_{q^2}^{\eta,\alpha}\phi(x) = (1-q^2)^{\beta} S_{q^2}^{\eta,\alpha+\beta}\phi(x) \quad (x \in \mathbb{R}_{q^2,+}). \tag{2.12}$$

Proposition 2.6. *Let α, β and η be complex numbers. If $\phi \in L_{q^2,\eta+\alpha-\gamma}(\mathbb{R}_{q^2,+})$ for some $\gamma \in \mathbb{C}$, $\Re(\gamma) > \max\{0, \Re(\alpha)\}$, then*

$$\mathcal{K}_{q^2}^{\eta,\alpha}\, S_{q^2}^{\eta+\alpha,\beta}\phi(x) = (1-q^2)^{\alpha} S_{q^2}^{\eta,\alpha+\beta}\phi(x) \quad (x \in \mathbb{R}_{q^2,+}). \tag{2.13}$$

Proposition 2.7. *Let α, β and η be complex numbers such that $\Re(\beta+\alpha) > 0$ and $\Re(2\eta+\alpha) > -1$. If $\phi \in L_{q^2,\eta}(\mathbb{R}_{q^2,+})$, then*

$$S_{q^2}^{\eta+\alpha,\beta}\, S_{q^2}^{\eta,\alpha}\phi(x) = (1-q^2)^{-\beta-\alpha} I_{q^2}^{\eta,\alpha+\beta}\phi(x) \quad (x \in \mathbb{R}_{q^2,+}). \tag{2.14}$$

Proposition 2.8. *Let α, β and η be complex numbers such that $\Re(\beta+\alpha) > 0$ and $\Re(2\eta+\alpha) > -1$. If $\phi \in L_{q^2,\eta+\alpha}(\mathbb{R}_{q^2,+})$, then*

$$S_{q^2}^{\eta,\alpha}\, S_{q^2}^{\eta+\alpha,\beta}\phi(x) = (1-q^2)^{-\beta-\alpha}\mathcal{K}_{q^2}^{\eta,\alpha+\beta}\phi(x) \quad (x \in \mathbb{R}_{q^2,+}). \tag{2.15}$$

3. Proof of Theorem 1.1

The goal of this section is to give two solutions of the triple q^2-integral equations (1.4)–(1.6) by reducing the system to a pair of q-simultaneous equations. Note that in this system, we have three range of ξ to consider, namely $\xi \in A_{q^2,a}$, $\xi \in A_{q^2,b} \cap B_{q^2,a}$ and $\xi \in B_{q^2,b}$. So, on the whole range we define two functions f and g as follows:

$$f = f_1 + f_2 + f_3 \quad \text{and} \quad g = g_1 + g_2 + g_3,$$

where

$$f \equiv f_1 \quad \text{on} \quad A_{q^2,a} \qquad \text{and} \qquad f_1 \equiv 0 \quad \text{otherwise},$$

$$f \equiv f_2 \quad \text{on} \quad A_{q^2,b} \cap B_{q^2,a} \quad \text{and} \quad f_2 \equiv 0 \quad \text{otherwise},$$
$$f \equiv f_3 \quad \text{on} \quad B_{q^2,b} \quad \text{and} \quad f_3 \equiv 0 \quad \text{otherwise},$$

with similar definitions for g_1, g_2 and g_3. Therefore, we can write the solution of the equation $g = I_{q^2}^{\eta,\alpha} f$ as follows:

$$g_1 = I_{q^2}^{\eta,\alpha} f_1 \quad \text{on} \quad A_{q^2,a},$$
$$g_2 = I_{q^2,0,a}^{\eta,\alpha} f_1 + I_{q^2,a}^{\eta,\alpha} f_2 \quad \text{on} \quad A_{q^2,b} \cap B_{q^2,a},$$
$$g_3 = I_{q^2,0,a}^{\eta,\alpha} f_1 + I_{q^2,a,b}^{\eta,\alpha} f_2 + I_{q^2,b}^{\eta,\alpha} f_3 \quad \text{on} \quad B_{q^2,b},$$

and the results of the equation $g = \mathcal{K}_{q^2}^{\eta,\alpha} f$ are

$$g_1 = \mathcal{K}_{q^2,a}^{\eta,\alpha} f_1 + \mathcal{K}_{q^2,a,b}^{\eta,\alpha} f_2 + \mathcal{K}_{q^2,b,\infty}^{\eta,\alpha} f_3 \quad \text{on} \quad A_{q^2,a},$$
$$g_2 = \mathcal{K}_{q^2,b}^{\eta,\alpha} f_2 + \mathcal{K}_{q^2,b,\infty}^{\eta,\alpha} f_3 \quad \text{on} \quad A_{q^2,b} \cap B_{q^2,a}$$
$$g_3 = \mathcal{K}_{q^2}^{\eta,\alpha} f_3 \quad \text{on} \quad B_{q^2,b}.$$

From (2.10) and above definitions, Eqs. (1.4)–(1.6) can be written as

$$(1-q^2)\, S_{q^2}^{\mu/2-\alpha,2\alpha} \psi(\xi) = f(\xi), \tag{3.1}$$

$$(1-q^2)\, S_{q^2}^{\mu/2,0} \psi(\xi) = g(\xi). \tag{3.2}$$

Here g_1, f_2 and g_3 are known, but g_2, f_1 and f_3 are unknown.

Following Sneddon [23], we take as a trial solution

$$\psi(\xi) = S_{q^2}^{\mu/2,-\alpha} h(\xi), \quad \xi \in \mathbb{R}_{q^2,+}. \tag{3.3}$$

Substituting (3.3) into (3.1) and (3.2) and then using Propositions 2.7 and 2.8 respectively, we obtain

$$f(\xi) = (1-q^2)^{1-\alpha} I_{q^2}^{\mu/2,\alpha} h(\xi), \tag{3.4}$$

$$g(\xi) = (1-q^2)^{1-\alpha} \mathcal{K}_{q^2}^{\mu/2,-\alpha} h(\xi). \tag{3.5}$$

Applying the inversion formulas (2.5) and (2.6), we get

$$h(\xi) = (1-q^2)^{\alpha-1}\, I_{q^2}^{\mu/2+\alpha,-\alpha} f(\xi), \tag{3.6}$$

$$h(\xi) = (1-q^2)^{\alpha-1}\, \mathcal{K}_{q^2}^{\mu/2-\alpha,\alpha} g(\xi). \tag{3.7}$$

The first solution.

From Eqs. (3.6) and (3.7), we obtain

$$I_{q^2}^{\mu/2+\alpha,-\alpha} f = \mathcal{K}_{q^2}^{\mu/2-\alpha,\alpha} g. \tag{3.8}$$

From (3.1) and (3.2) and then using (2.12) and (2.13), we get

$$S_{q^2}^{\mu/2,\alpha} \psi := I_{q^2}^{\mu/2,\alpha} g = \mathcal{K}_{q^2}^{\mu/2,-\alpha} f. \tag{3.9}$$

Now, taking (3.8) on $B_{q^2,b}$, we get

$$I_{q^2,0,a}^{\mu/2+\alpha,-\alpha} f_1 + I_{q^2,a,b}^{\mu/2+\alpha,-\alpha} f_2 + I_{q^2,b}^{\mu/2+\alpha,-\alpha} f_3 = \mathcal{K}_{q^2}^{\mu/2-\alpha,\alpha} g_3, \quad (x > b).$$

Therefore,

$$f_3 = \left(I^{\mu/2+\alpha,-\alpha}_{q^2,b}\right)^{-1} \mathcal{K}^{\mu/2-\alpha,\alpha}_{q^2}\, g_3 - {}_bL^{\mu/2+\alpha,-\alpha}_{q^2,0,a}\, f_1 - {}_bL^{\mu/2+\alpha,-\alpha}_{q^2,a,b}\, f_2. \tag{3.10}$$

Similarly, taking Eq. (3.18) on $A_{q^2,a}$, we obtain

$$f_1 = \left(\mathcal{K}^{\mu/2,-\alpha}_{q^2,a}\right)^{-1} I^{\mu/2,\alpha}_{q^2}\, g_1 - {}_aM^{\mu/2,-\alpha}_{q^2,a,b}\, f_2 - {}_aM^{\mu/2,-\alpha}_{q^2,b,\infty}\, f_3. \tag{3.11}$$

Note that Eqs. (3.10) and (3.11) form a pair of q-simultaneous equations for f_1 and f_3. Hence ψ follows by applying the inverse pair of q-Hankel transformation on (1.5).

Example 1. Assume that g_1 and g_3 are zeros. Our aim to find the function h (notice, $h = h_1 + h_2 + h_3$). Since $g_3 \equiv 0$, the result of Eq. (3.7) on $B_{q^2,b}$ is $h_3 \equiv 0$.

Evaluating (3.4) on $A_{q^2,a}$, we get

$$f_1 = (1-q^2)^{1-\alpha} I^{\mu/2,\alpha}_{q^2} h_1. \tag{3.12}$$

Evaluating (3.6) on $A_{q^2,b} \cap B_{q^2,a}$, we get

$$h_2 = (1-q^2)^{\alpha-1}\Big(I^{\mu/2+\alpha,-\alpha}_{q^2,0,a}\, f_1 + I^{\mu/2+\alpha,-\alpha}_{q^2,a}\, f_2\Big). \tag{3.13}$$

Substituting from (3.12) into (3.13) and then using (2.8), we get

$$h_2 = -\,{}_aL^{\mu/2,\alpha}_{q^2,0,a}\, h_1 + (1-q^2)^{\alpha-1}\, I^{\mu/2+\alpha,-\alpha}_{q^2,a}\, f_2. \tag{3.14}$$

Evaluating (3.5) on $A_{q^2,b} \cap B_{q^2,a}$ and (3.7) on $A_{q^2,a}$, we have the following equations:

$$h_1 = (1-q^2)^{\alpha-1}\, \mathcal{K}^{\mu/2-\alpha,\alpha}_{q^2,a,b}\, g_2. \tag{3.15}$$

$$g_2 = (1-q^2)^{1-\alpha}\, \mathcal{K}^{\mu/2,-\alpha}_{q^2,b} h_2. \tag{3.16}$$

Substituting from (3.16) into (3.15) and then using (2.9), we obtain

$$h_1 = -\,{}_aM^{\mu/2,-\alpha}_{q^2,a,b}\, h_2. \tag{3.17}$$

Now, substituting from (3.17) into (3.14), we obtain

$$h_2 = {}_aL^{\mu/2,\alpha}_{q^2,0,a}\, {}_aM^{\mu/2,-\alpha}_{q^2,a,b}\, h_2 + (1-q^2)^{\alpha-1}\, I^{\mu/2+\alpha,-\alpha}_{q^2,a}\, f_2. \tag{3.18}$$

Thus, $h_2(\xi)$ satisfies the Fredholm q-integral equation of the second kind which can then be solved numerically.

Remark 3.1. Example 1 is a q-type analogue of the problem by Cooke [6] and Williams [25].

The second solution.

Since the triple q^2-integral equations (1.4)–(1.6) are linear in ψ, we may assume that $\psi := \psi + \overline{\psi}$ and

$$f_2 = f_{2a} + \overline{f}_{2b},$$

where f_{2a} is defined on $\mathbb{R}_{q^2,+} \setminus B_{q^2,b}$ and $\overline{f}_{2b}$ is defined on $\mathbb{R}_{q^2,+} \setminus A_{q^2,a}$.

Note that f_2 is defined only on $A_{q^2,b} \cap B_{q^2,a}$, but in this problem, following Cooke [7], we need the above splitting. Therefore, we can define $Q\psi$ and $Q\overline{\psi}$ as follows:

$$Q\psi = \begin{cases} f_{2a}(\xi), & \xi \in \mathbb{R}_{q^2,+} \setminus B_{q^2,b}, \\ f_3(\xi), & \xi \in B_{q^2,b}. \end{cases}$$

$$Q\overline{\psi} = \begin{cases} \overline{f}_1(\xi), & \xi \in A_{q^2,a}, \\ \overline{f}_{2b}(\xi), & \xi \in \mathbb{R}_{q^2,+} \setminus A_{q^2,a}. \end{cases}$$

Similarly, we split up g_1 and g_3 such that

$$P\psi = \begin{cases} g_{1a}(\xi), & \xi \in \mathbb{R}_{q^2,+} \setminus B_{q^2,b}, \\ g_3(\xi), & \xi \in B_{q^2,b}. \end{cases}$$

$$Q\overline{\psi} = \begin{cases} \overline{g}_1(\xi), & \xi \in A_{q^2,a}, \\ \overline{g}_{3b}(\xi), & \xi \in \mathbb{R}_{q^2,+} \setminus A_{q^2,a}. \end{cases}$$

That is,

$$g_1(\xi) = g_{1a}(\xi) + \overline{g}_1(\xi), \quad \xi \in A_{q^2,a}, \tag{3.19}$$

$$g_3(\xi) = g_3(\xi) + \overline{g}_{3b}(\xi), \quad \xi \in B_{q^2,b}. \tag{3.20}$$

Our aim is to find the unknown functions g_{1a} and $\overline{g_{3b}}$. We have the following:

$$I_{q^2}^{\mu/2+\alpha,-\alpha} f = \mathcal{K}_{q^2}^{\mu/2-\alpha,\alpha} g, \tag{3.21}$$

$$I_{q^2}^{\mu/2+\alpha,-\alpha} \overline{f} = \mathcal{K}_{q^2}^{\mu/2-\alpha,\alpha} \overline{g}, \tag{3.22}$$

$$\mathcal{K}_{q^2}^{\mu/2,-\alpha} f = I_{q^2}^{\mu/2,\alpha} g, \tag{3.23}$$

$$\mathcal{K}_{q^2}^{\mu/2,-\alpha} \overline{f} = I_{q^2}^{\mu/2,\alpha} \overline{g}. \tag{3.24}$$

Evaluating (3.21) on $\mathbb{R}_{q^2,+} \setminus B_{q^2,b}$, we obtain

$$I_{q^2}^{\mu/2+\alpha,-\alpha} f_{2a} = \mathcal{K}_{q^2,b}^{\mu/2-\alpha,\alpha} g_{1a} + \mathcal{K}_{q^2,b,\infty}^{\mu/2-\alpha,\alpha} g_3, \quad 0 < x < b.$$

This implies

$$g_{1a} = \mathcal{K}_{q^2,b}^{\mu/2,-\alpha} I_{q^2}^{\mu/2+\alpha,-\alpha} f_{2a} - {}_bM_{q^2,b,\infty}^{\mu/2,-\alpha} (g_3 + \overline{g}_{3b}). \tag{3.25}$$

Similarly, taking (3.23) on $\mathbb{R}_{q^2,+} \setminus A_{q^2,a}$ and using (3.19), we obtain

$$\overline{g}_{3b} = I_{q^2,a}^{\mu/2+\alpha,-\alpha} \mathcal{K}_{q^2}^{\mu/2,-\alpha} \overline{f}_{2b} - {}_aL_{q^2,0,a}^{\mu/2,\alpha} (g_3 + \overline{g}_{3b}), \quad a < x. \tag{3.26}$$

Eqs. (3.25) and (3.26) form a pair of q-simultaneous equations for g_{1a} and $\overline{g}_{3b}$.

4. Solving system of triple q^2-integral equations with general kernel

The goal of this section is to solve the following triple q-integral equations:

$$\int_0^\infty \psi(\rho) J_\mu(\sqrt{\rho\xi}; q^2)\, d_{q^2}\rho = g_1(\xi), \quad \xi \in A_{q^2,a},$$

$$\xi^{-\alpha} \int_0^\infty \rho^{-\alpha} [1 + w(\rho)]\, \psi(\rho) J_\mu(\sqrt{\rho\xi}; q^2)\, d_{q^2}\rho = f_2(\xi), \quad \xi \in A_{q^2,b} \cap B_{q^2,a},$$

$$\int_0^\infty \psi(\rho) J_\mu(\sqrt{\rho\xi}; q^2)\, d_{q^2}\rho = g_3(\xi), \quad \xi \in B_{q^2,b},$$

where $0 < a < b < \infty$, and α, μ are complex numbers satisfying

$$\Re(\mu) > -1, \text{ and } 0 < \Re(\alpha) < 1,$$

g_1, f_2 and g_3 are known functions, w is a non-negative bounded function defined on $\mathbb{R}_{q,+}$, and

$$\psi \in L_{q^2,\frac{\mu}{2}-\alpha}(\mathbb{R}_{q^2,+}) \cap L_{q^2,\mu}(\mathbb{R}_{q^2,+}),$$

which is an unknown function to be determined.

By the same argument in Section 3, we obtain

$$(1-q^2)\, S_{q^2}^{\mu/2-\alpha,2\alpha} [1 + w(\rho)]\, \psi(\xi) = f(\xi),$$
$$(1-q^2)\, S_{q^2}^{\mu/2,0} \psi(\xi) = g(\xi).$$

Now, we will take as trial solution

$$\psi(\xi) = S_{q^2}^{\mu/2,-\alpha} h(\xi), \quad \xi \in \mathbb{R}_{q^2,+},$$

and we can use Proposition 2.4 to obtain

$$S_{q^2}^{\mu/2-\alpha,\alpha} \psi(\xi) = h(\xi), \quad \xi \in \mathbb{R}_{q^2,+}.$$

Then, we get

$$f(\xi) = (1-q^2)^{1-\alpha} I_{q^2}^{\mu/2,\alpha} h(\xi) + (1-q^2) S_{q^2}^{\mu/2-\alpha,2\alpha} w(\xi)\, S_{q^2}^{\mu/2,-\alpha} h(\xi), \tag{4.1}$$

$$g(\xi) = (1-q^2)^{1-\alpha} \mathcal{K}_{q^2}^{\mu/2,-\alpha} h(\xi). \tag{4.2}$$

From (4.1) and using Eq. (2.12), we get

$$h = (1-q^2)^{\alpha-1} I_{q^2}^{\mu/2+\alpha,-\alpha} f - (1-q^2)^\alpha S_{q^2}^{\mu/2-\alpha,\alpha} w\, S_{q^2}^{\mu/2,-\alpha} h. \tag{4.3}$$

Inverting (4.2), we get

$$h = (1-q^2)^{\alpha-1} \mathcal{K}_{q^2}^{\mu/2-\alpha,\alpha} g. \tag{4.4}$$

Now, we put

$$E(\xi) = (1-q^2)^\alpha S_{q^2}^{\mu/2-\alpha,\alpha} w\, S_{q^2}^{\mu/2,-\alpha} h.$$

That is,

$$E(\xi) = \frac{\xi^{-\alpha/2}}{(1-q^2)^{2-\alpha}} \int_0^\infty w(u) J_{\mu-\alpha}(\sqrt{u\xi}; q^2) \int_0^\infty y^{\alpha/2} h(y) J_{\mu-\alpha}(\sqrt{uy}; q^2)\, d_{q^2}y\, d_{q^2}u.$$

Therefore, Eq. (4.3) may be written as

$$h + E(\xi) = (1-q^2)^{\alpha-1} I_{q^2}^{\mu/2+\alpha,-\alpha} f. \tag{4.5}$$

Evaluating (4.5) on $A_{q^2,a}$, we obtain

$$h_1 + E(\xi) = (1-q^2)^{\alpha-1} k_1, \tag{4.6}$$

$$f_1 = I_{q^2}^{\mu/2,\alpha} k_1. \tag{4.7}$$

Again, taking (4.5) on $A_{q^2,b} \cap B_{q^2,a}$ and then, using Eqs. (4.7) and (2.8) we get

$$h_2 + E(\xi) = (1-q^2)^{\alpha-1}\left(-\,{}_aL^{\mu/2,\alpha}_{q^2,0,a}\,k_1 + I^{\mu/2+\alpha,-\alpha}_{q^2,a}\,f_2\right). \tag{4.8}$$

Now, evaluating (4.4) on $A_{q^2,a}$, $A_{q^2,b} \cap B_{q^2,a}$ and $B_{q^2,b}$ respectively, we obtain the following equations:

$$h_1 = (1-q^2)^{\alpha-1}\left(\mathcal{K}^{\mu/2-\alpha,\alpha}_{q^2,a}\,g_1 + \mathcal{K}^{\mu/2-\alpha,\alpha}_{q^2,a,b}\,g_2 + \mathcal{K}^{\mu/2-\alpha,\alpha}_{q^2,b,\infty}\,g_3\right), \tag{4.9}$$

$$h_2 = (1-q^2)^{\alpha-1}\left(\mathcal{K}^{\mu/2-\alpha,\alpha}_{q^2,b}\,g_2 + \mathcal{K}^{\mu/2-\alpha,\alpha}_{q^2,b,\infty}\,g_3\right), \tag{4.10}$$

$$h_3 = \mathcal{K}^{\mu/2-\alpha,\alpha}_{q^2}\,g_3. \tag{4.11}$$

From (4.10), we have

$$g_2 = (1-q^2)^{1-\alpha}\left(\mathcal{K}^{\mu/2,-\alpha}_{q^2,b}\,h_2 - {}_bM^{\mu/2-\alpha,\alpha}_{q^2,b,\infty}\,g_3\right). \tag{4.12}$$

Substituting from (4.12) into (4.9) and using (2.9), we find

$$\begin{aligned} h_1 = {} & (1-q^2)^{\alpha-1}\left(\mathcal{K}^{\mu/2-\alpha,\alpha}_{q^2,a}\,g_1 + \mathcal{K}^{\mu/2-\alpha,\alpha}_{q^2,b,\infty}\,g_3\right) \\ & - {}_aM^{\mu/2,-\alpha}_{q^2,a,b}\left(h_2 - \mathcal{K}^{\mu/2-\alpha,\alpha}_{q^2}\,g_3\right), \end{aligned} \tag{4.13}$$

where g_1 and g_3 are known functions. Thus, we have the following result which completes the solution of our problem.

Proposition 4.1. *For $\xi \in \mathbb{R}_{q^2,+}$, h_1, h_2, h_3 and k_1 satisfy the following system of Fredholm q-integral equation:*

$$h_1 + E(\xi) = (1-q^2)^{\alpha-1}k_1,$$

$$h_2 + E(\xi) = (1-q^2)^{\alpha-1}\left(-{}_aL^{\mu/2,\alpha}_{q^2,0,a}\,k_1 + I^{\mu/2+\alpha,-\alpha}_{q^2,a}\,f_2\right),$$

$$h_1 = (1-q^2)^{\alpha-1}\left(\mathcal{K}^{\mu/2-\alpha,\alpha}_{q^2,a}\,g_1 + \mathcal{K}^{\mu/2-\alpha,\alpha}_{q^2,b,\infty}\,g_3\right) - {}_aM^{\mu/2,-\alpha}_{q^2,a,b}\left(h_2 - \mathcal{K}^{\mu/2-\alpha,\alpha}_{q^2}\,g_3\right),$$

$$h_3 = \mathcal{K}^{\mu/2-\alpha,\alpha}_{q^2}\,g_3.$$

Remark 4.2. In the above proposition, if $g_3 \equiv 0$, then $h_3 \equiv 0$ and the result is reduced to three equations.

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