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Some results on modules satisfying S-strong *accr**

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Abstract. The rings considered in this article are commutative with identity. Modules are assumed to be unitary. Let R be a ring and let S be a multiplicatively closed subset of R. We say that a module M over R *satisfies* S*- strong accr** if for every submodule N of M and for every sequence $< r_n >$ of elements of R, the ascending sequence of submodules $(N :_M r_1) \subseteq (N :_M r_1r_2) \subseteq (N :_M r_1r_2r_3) \subseteq \cdots$ is S-stationary. That is, there exist $k \in \mathbb{N}$ and $s \in S$ such that $s(N :_M r_1 \cdots r_n) \subseteq (N :_M r_1 \cdots r_k)$ for all $n \geq k$. We say that a ring R *satisfies* S*- strong accr** if R regarded as a module over R satisfies S-strong *accr**. The aim of this article is to study some basic properties of rings and modules satisfying S-strong *accr**.

Keywords: Strong *accr**; S-strong *accr**; Perfect ring; S-n-acc

Mathematics Subject Classification: 13A15

1. INTRODUCTION

The rings considered in this article are commutative with identity. Modules are assumed to be unitary. Let R be a ring. If S is a multiplicatively closed subset of R , then we assume that $0 \notin S$ and $1 \in S$. We use m.c. set to denote multiplicatively closed set. We use the abbreviation f.g. for finitely generated. Let M be a module over a ring R and let S be a m.c. subset of R. Recall from [2] that M is said to be *S-finite* if there exist $s \in S$ and a f.g. submodule N of M such that $sM \subseteq N$ and M is said to be *S-Noetherian* if any

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submodule of M is S-finite. A ring R is said to be *S-Noetherian* if R regarded as a module over R is S-Noetherian. A very interesting and inspiring investigation on S-Noetherian rings and S-Noetherian modules has been carried out in [2] and the article [2] contains S-variant of several properties of Noetherian modules to S-Noetherian modules. This article is motivated by the interesting theorems proved by D.D. Anderson and T. Dumitrescu in [2]. To justify this statement, we mention some results that were proved in [2]. Let S be a m.c. subset of a ring R and let M be a module over R. Let $\psi : M \to S^{-1}M$ denote the usual R-homomorphism defined by $\psi(m) = \frac{m}{1}$. For any submodule N of M, $\psi^{-1}(S^{-1}N)$ is called the *saturation of N with respect to S* and is denoted by $Sat_S(N)$. It was shown in [2, Proposition 2(f)] that a ring R is S-Noetherian if and only if $S^{-1}R$ is Noetherian and for every f.g. ideal I of R, $Sat_S(I) = (I :_R s)$ for some $s \in S$. Let M be a S-finite module over R. In [2, Proposition 4], it was proved that M is S-Noetherian if and only if the submodules of the form PM are S-finite for each prime ideal P of R disjoint from S and it was deduced in [2, Corollary 5] that a ring R is S-Noetherian if and only if every prime ideal of R disjoint from S is S-finite and this is the S-variant of Cohen's Theorem. Let $A \subseteq B$ be a ring extension and $S \subseteq A$ be a m.c. subset such that B is a S-finite A-module. It was shown in [2, Corollary 7] that if B is S-Noetherian, then so is A and this is the S-variant of Eakin–Nagata Theorem. Recall from [2, page 4411] that a m.c. subset S of R is said to be *anti-Archimedean* if $(\cap_{n=1}^{\infty} s^n R) \cap S \neq \emptyset$ for every $s \in S$. Let S be an anti-Archimedean m.c. subset of a ring R. It was proved in [2, Proposition 9] that if R is S-Noetherian, then so is the polynomial ring $R[X_1, \ldots, X_n]$ and this is the S-variant of Hilbert Basis Theorem and it was shown in [2, Proposition 10] that if S consists of nonzero-divisors and if R is S-Noetherian, then so is the power series ring $R[[X_1, \ldots, X_n]]$.

Let M be a module over a ring R. Recall from [7] that M *satisfies accr* (respectively, *satisfies accr**) if the ascending chain of submodules of the form $(N :_M B) \subseteq (N :_M B^2) \subseteq (N :_M B^3) \subseteq \cdots$ terminates for every submodule N of M and every f.g. (respectively, principal) ideal B of R. A ring R is said to *satisfy accr* (respectively, *satisfy accr**) if R regarded as a module over R satisfies accr (respectively, *accr**). It is known that a module M over a ring R satisfies accr if and only if M satisfies *accr** [7, Theorem 1]. In [7,8], Chin-Pi Lu has shown that many important properties of Noetherian modules are possessed by modules satisfying accr.

Let R be a ring and let S be a m.c. subset of R. Inspired by the articles [2,7,8], H. Ahmed and H. Sana introduced and investigated the concept of modules satisfying S-accr and *S-accr** in [1]. Let M be a module over R. Recall from [1] that an ascending sequence of submodules $N_1 \subseteq N_2 \subseteq N_3 \subseteq \cdots$ of M is *S-stationary* if there exist $k \in \mathbb{N}$ and $s \in S$ such that $sN_n \subseteq N_k$ for all $n \geq k$. Recall from [1, Definition 3.1] that *M satisfies S-accr* (respectively, *satisfies S-accr**) if the ascending sequence of submodules of the form $(N :_M B) \subseteq (N :_M B^2) \subseteq (N :_M B^3) \subseteq \cdots$ is S-stationary for any submodule N of M and any f.g. (respectively, principal) ideal B of R. A ring R is said to *satisfy S-accr* (respectively, *satisfy S-accr**) if R regarded as a module over R satisfies S-accr (respectively, *S-accr**). Several results from [7,8] on modules satisfying accr have been extended in [1] to modules satisfying S-accr.

Let M be a module over a ring R. We say that M *satisfies* (C) if the ascending sequence of submodules of the form $(N :_M r_1) \subseteq (N :_M r_1r_2) \subseteq (N :_M r_1r_2r_3) \subseteq \cdots$ terminates for any submodule N of M and for any sequence $\langle r_n \rangle$ of elements of R [12]. It is clear that if a module M over a ring R satisfies (C), then M satisfies *accr**. Hence, it is convenient to replace the condition (C) by strong-*accr**. We say that M *satisfies strong accr** if M

satisfies (C). We say that R *satisfies strong* $accr^*$ if R regarded as a module over R satisfies strong $accr^*$. A study was carried out on rings and modules satisfying strong $accr^*$ in [12].

Let S be a m.c. subset of a ring R. It was shown in [1, Proposition 3.1] that for any R-module M, the properties S-accr and S-$accr^*$ are equivalent. It was proved in [1, Lemma 3.6] that R satisfies S-accr if and only if the R-module R^n satisfies S-accr for each $n \in \mathbb{N}$. Let M be a f.g. module over R. In [1, Theorem 3.3], it was shown that if R satisfies S-accr, then so does M. We denote the polynomial ring in one variable X over a ring R by $R[X]$. If S is finite, then it was proved in [1, Theorem 3.4] that $R[X]$ satisfies S-accr if and only if R is S-Noetherian. Motivated by the work on S-accr modules in [1], in this article, we introduce the concept of modules satisfying S-strong $accr^*$ and try to investigate some properties of modules satisfying S-strong $accr^*$. Let R be a ring and let S be a m.c. subset of R. We say that a module M over R *satisfies* S*-strong* $accr^*$ if for any submodule N of M and for any sequence $\langle r_n \rangle$ of elements of R, the ascending sequence of submodules of the form $(N :_M r_1) \subseteq (N :_M r_1r_2) \subseteq (N :_M r_1r_2r_3) \subseteq \cdots$ is S-stationary. We say that R *satisfies* S*-strong* $accr^*$ if R regarded as a module over R satisfies S-strong $accr^*$.

In Section 2 of this article, we prove some basic properties of modules satisfying S-strong $accr^*$. Let S be a m.c. subset of a ring R and let M be a module over R. The main result proved in Section 2 is Theorem 2.7 in which necessary and sufficient conditions are determined for a module M over a ring R to satisfy S-strong $accr^*$, where S is a countable m.c. subset of R. For a countable m.c. subset S of R, in Theorem 2.8, the question of when every module over R satisfies S-strong $accr^*$ is answered. Let $n \geq 1$. Inspired by the work on rings and modules satisfying n-acc and pan-acc by W. Heinzer and D. Lantz in [6] and by G. Renault in [9], the concept of S-n-acc and S-pan-acc are introduced and it is shown that for a countable m.c. subset S of a ring R, if every module over R satisfies S-strong $accr^*$, then every module over R satisfies S-pan-acc. Examples are given to illustrate some of the results proved in Section 2 (see Examples 2.3, 2.5, 2.9, 2.11, and 2.13).

In Section 3 of this article, some more properties of modules satisfying S-strong $accr^*$ are proved. Let S be a m.c. subset of an integral domain R such that R satisfies S-strong $accr^*$. In Proposition 3.4, the problem of when a free module F over R satisfies S-strong $accr^*$ is answered. Let S be a countable m.c. subset of R. It is shown in Theorem 3.6 that the polynomial ring $R[X]$ satisfies S-strong $accr^*$ if and only if $R[X]$ is S-Noetherian.

Let R be a ring. The Krull dimension of R is simply referred to as the dimension of R and is denoted by the notation $dim R$. We denote the set of all units of R by $U(R)$. Whenever a set A is a subset of a set B and $A \neq B$, we denote it symbolically by the notation $A \subset B$.

2. Some basic properties of modules satisfying S-strong $accr^*$

Let M be a module over a ring R. If M satisfies strong $accr^*$, then it is clear that M satisfies $accr^*$ and so, M satisfies accr [7, Theorem 1]. In Remark 2.1(i), we mention an example of a module M over $\mathbb{Z}$ such that M satisfies *accr* but M does not satisfy strong $accr^*$.

Remark 2.1. (i) Let us denote the set of all prime numbers by $\mathbb{P}$. Let $M = \bigoplus_{p \in \mathbb{P}} \frac{\mathbb{Z}}{p\mathbb{Z}}$. We know from [7, Example 1] that the $\mathbb{Z}$-module M satisfies accr. It was shown in [12, see page 164] that M does not satisfy strong $accr^*$.

(ii) Let M be a module over a ring R. Let S be a m.c. subset of R. If M satisfies strong $accr^*$, then M satisfies S-strong $accr^*$. In Example 2.13, we provide an example of a domain R and a m.c. subset S of R such that R satisfies S-strong $accr^*$ but R does not satisfy strong $accr^*$. $\square$

Let M be a module over a ring R and let S be a m.c. subset of R. If M is S- Noetherian, then it is not hard to show that any ascending sequence of submodules of M is S-stationary. Hence, we obtain that M satisfies S-strong $accr^*$. We provide Example 2.3 to illustrate that a module satisfying S-strong $accr^*$ can fail to be S-Noetherian.

Lemma 2.2. *Let V be a vector space over a field K. Then V satisfies strong $accr^*$.*

Proof. Let W be any subspace of V and let $\alpha \in K$. Note that $(W :_V \alpha) = V$ if $\alpha = 0$ and it is equal to W if $\alpha \neq 0$. Let $\langle \alpha_n \rangle$ be any sequence of elements of K. If $\alpha_k = 0$ for some $k \in \mathbb{N}$, then for all $n \geq k$, $(W :_V \alpha_1 \cdots \alpha_n) = (W :_V \alpha_1 \cdots \alpha_k) = V$. If $\alpha_i \neq 0$ for all $i \in \mathbb{N}$, then $(W :_V \alpha_1 \cdots \alpha_i) = (W :_V \alpha_1 \cdots \alpha_j) = W$ for all $i, j \in \mathbb{N}$. This shows that V satisfies strong $accr^*$. $\square$

Example 2.3. Let V be an infinite dimensional vector space over a field K. Then for any m.c. subset S of K, V satisfies S-strong $accr^*$ but V is not S-Noetherian.

Proof. We know from Lemma 2.2 that V satisfies strong $accr^*$ and so, V satisfies S-strong $accr^*$ for any m.c. subset S of K. Let S be any m.c. subset of K. Note that $S \subseteq K \backslash \{0\} = U(K)$. Hence, for any subspace W of V and for any $s \in S$, $sW = W$. Since we are assuming that $dim_K V$ is infinite, there exists a strictly ascending sequence of subspaces $W_1 \subset W_2 \subset W_3 \subset \cdots$ of V. It is clear that there exist no $k \in \mathbb{N}$ and $s \in S$ such that $sW_n \subseteq W_k$ for all $n \geq k$. Therefore, V is not S-Noetherian for any m.c. subset S of K. $\square$

Lemma 2.4. *Let M be a module over a ring R and let S be a m.c. subset of R. If M satisfies S- strong $accr^*$, then the $S^{-1}R$-module $S^{-1}M$ satisfies strong $accr^*$.*

Proof. Let W be any $S^{-1}R$-submodule of $S^{-1}M$. Let $\langle x_n \rangle$ be e sequence of elements of $S^{-1}R$. Note that $W = S^{-1}N$ for some R-submodule N of M and for each $n \in \mathbb{N}$, let $x_n = \frac{r_n}{s_n}$ for some $r_n \in R$ and $s_n \in S$. Since M satisfies S-strong $accr^*$ by hypothesis, we obtain that the ascending sequence of submodules $(N :_M r_1) \subseteq (N :_M r_1 r_2) \subseteq (N :_M r_1 r_2 r_3) \subseteq \cdots$ of M is S-stationary. Hence, there exist $s \in S$ and $k \in \mathbb{N}$ such that $s(N :_M r_1 \cdots r_n) \subseteq (N :_M r_1 \cdots r_k)$ for all $n \geq k$. This implies that $S^{-1}(s(N :_M r_1 \cdots r_n)) \subseteq S^{-1}(N :_M r_1 \cdots r_k) \subseteq S^{-1}(N :_M r_1 \cdots r_n)$ for all $n \geq k$ and so, $(S^{-1}N :_{S^{-1}M} \frac{r_1}{s_1} \cdots \frac{r_n}{s_n}) = (S^{-1}N :_{S^{-1}M} \frac{r_1}{s_1} \cdots \frac{r_k}{s_k})$ for all $n \geq k$. Thus there exists $k \in \mathbb{N}$ such that $(W :_{S^{-1}M} x_1 \cdots x_n) = (W :_{S^{-1}M} x_1 \cdots x_k)$ for all $n \geq k$. This shows that the $S^{-1}R$-module $S^{-1}M$ satisfies strong $accr^*$. $\square$

We provide Example 2.5 to illustrate that the converse of Lemma 2.4 can fail to hold.

Example 2.5. Let us denote the set of all prime numbers by $\mathbb{P}$. Let $\mathbb{P} = \{p_1 = 2 < p_2 = 3 < p_3 < p_4 < \cdots\}$. Let M be the $\mathbb{Z}$-module given by $M = \bigoplus_{p \in \mathbb{P}} \frac{\mathbb{Z}}{p\mathbb{Z}}$. Let $S = \mathbb{Z} \backslash 2\mathbb{Z}$. Then the $S^{-1}\mathbb{Z}$-module $S^{-1}M$ satisfies strong $accr^*$ but M does not satisfy S-strong $accr^*$.

Proof. Let $p \in \mathbb{P}$. It is not hard to verify that $M_{p\mathbb{Z}} \cong \frac{\mathbb{Z}_{p\mathbb{Z}}}{p\mathbb{Z}_{p\mathbb{Z}}}$ as $\mathbb{Z}_{p\mathbb{Z}}$-modules. Let $S = \mathbb{Z}\backslash 2\mathbb{Z}$. Note that S is a m.c. subset of $\mathbb{Z}$ and as $M_{2\mathbb{Z}}$ is a finite $\mathbb{Z}_{2\mathbb{Z}}$-module, we obtain that the $\mathbb{Z}_{2\mathbb{Z}}$-module $S^{-1}M = M_{2\mathbb{Z}}$ satisfies strong $accr^*$. We now verify that the $\mathbb{Z}$-module M does not satisfy S-strong $accr^*$. Let us denote the zero submodule of M simply by $(\overline{0}, \overline{0}, \overline{0}, \ldots)$. We know from [12, page 164] that $((\overline{0}, \overline{0}, \overline{0}, \ldots) :_M p_1) \subset ((\overline{0}, \overline{0}, \overline{0}, \ldots) :_M p_1p_2) \subset ((\overline{0}, \overline{0}, \overline{0}, \ldots) :_M p_1p_2p_3) \subset \cdots$ is a strictly ascending sequence of submodules of M. We claim that this sequence is not S-stationary. Suppose that the above sequence of submodules of M is S-stationary. Then there exist $k \in \mathbb{N}$ and $s \in S$ such that $s((\overline{0}, \overline{0}, \overline{0}, \ldots) :_M p_1 \ldots p_n) \subseteq ((\overline{0}, \overline{0}, \overline{0}, \ldots) :_M p_1 \cdots p_k)$ for all $n \geq k$. We can assume that $s > 0$. It is clear that $s \neq 1$. As $s \in S = \mathbb{Z}\backslash 2\mathbb{Z}$, it follows that there exist distinct $p_{i_1}, \ldots, p_{i_t} \in \mathbb{P}\backslash\{2\}$ and positive integers $n_1, \ldots, n_t$ such that $s = \prod_{j=1}^{t} p_{i_j}^{n_j}$. Observe that $s((\overline{0}, \overline{0}, \overline{0}, \ldots,) :_M p_1p_2 \cdots p_{k+i_1+\cdots+i_t}) = \bigoplus_{j\in T} \frac{\mathbb{Z}}{p_j\mathbb{Z}}$, where $T = \{1, 2, \ldots, k + i_1 + \cdots + i_t\}\backslash\{i_1, \ldots, i_t\}$, $((\overline{0}, \overline{0}, \overline{0}, \ldots) :_M p_1 \cdots p_k) = \frac{\mathbb{Z}}{p_1\mathbb{Z}} \oplus \cdots \oplus \frac{\mathbb{Z}}{p_k\mathbb{Z}}$ and so, we get that $\frac{\mathbb{Z}}{p_{k+i_1+\cdots+i_t}\mathbb{Z}} \subseteq \frac{\mathbb{Z}}{p_1\mathbb{Z}} \oplus \cdots \oplus \frac{\mathbb{Z}}{p_k\mathbb{Z}}$. This is impossible. Therefore, the ascending sequence of submodules $((\overline{0}, \overline{0}, \overline{0}, \ldots) :_M p_1) \subset ((\overline{0}, \overline{0}, \overline{0}, \ldots) :_M p_1p_2) \subset ((\overline{0}, \overline{0}, \overline{0}, \ldots) :_M p_1p_2p_3) \subset \cdots$ is not S-stationary. This shows that the $\mathbb{Z}$-module M does not satisfy S-strong $accr^*$. □

In Lemma 2.6, we provide a sufficient condition on a module M over a ring R under which the converse of Lemma 2.4 holds.

Lemma 2.6. *Let M be a module over a ring R and let S be a m.c. subset of R. If the $S^{-1}R$-module $S^{-1}M$ satisfies strong $accr^*$ and if for any submodule N of M, there exists $s \in S$ (depending on N) such that $Sat_S(N) = (N :_M s)$, then M satisfies S-strong $accr^*$.*

Proof. Let N be a submodule of M and let $\langle r_n \rangle$ be a sequence of elements of R. We prove that the ascending sequence $(N :_M r_1) \subseteq (N :_M r_1r_2) \subseteq (N :_M r_1r_2r_3) \subseteq \cdots$ of submodules of M is S-stationary. For each $n \in \mathbb{N}$, let us denote $\frac{r_n}{1}$ by x_n. Then $\langle x_n \rangle$ is a sequence of elements of $S^{-1}R$. Consider the ascending sequence of submodules of $S^{-1}M$ given by $(S^{-1}N :_{S^{-1}M} x_1) \subseteq (S^{-1}N :_{S^{-1}M} x_1x_2) \subseteq (S^{-1}N :_{S^{-1}M} x_1x_2x_3) \subseteq \cdots$. Since the $S^{-1}R$-module $S^{-1}M$ satisfies strong $accr^*$, there exists $k \in \mathbb{N}$ such that for all $n \geq k$, $(S^{-1}N :_{S^{-1}M} x_1 \cdots x_n) = (S^{-1}N :_{S^{-1}M} x_1 \cdots x_k)$. By hypothesis, there exists $s \in S$ such that $Sat_S(N) = (N :_M s)$. Let $n \geq k$. Let $m \in (N :_M r_1 \cdots r_n)$. Then $\frac{m}{1} \in (S^{-1}N :_{S^{-1}M} x_1 \cdots x_n) = (S^{-1}N :_{S^{-1}M} x_1 \cdots x_k)$. This implies that $s'm \in (N :_M r_1 \cdots r_k)$ for some $s' \in S$ and so, $s'r_1 \cdots r_km \in N$. Therefore, $r_1 \cdots r_km \in Sat_S(N) = (N :_M s)$. This proves that $s(N :_M r_1 \cdots r_n) \subseteq (N :_M r_1 \cdots r_k)$ for all $n \geq k$. Therefore, we obtain that M satisfies S-strong $accr^*$. □

Let S be a countable m.c. subset of a ring R. With the help of Lemmas 2.4 and 2.6, in Theorem 2.7, we provide a necessary and sufficient condition in order that a module M over R satisfies S-strong $accr^*$.

Theorem 2.7. *Let M be a module over a ring R. Let S be a countable m.c. subset of R. The following statements are equivalent:*

(i) M satisfies S-strong $accr^*$.

(ii) The $S^{-1}R$-module $S^{-1}M$ satisfies strong $accr^*$ and for any submodule N of M, there exists $s \in S$(depending on N) such that $Sat_S(N) = (N :_M s)$.

Proof. $(i) \Rightarrow (ii)$ Assume that M satisfies S-strong $accr^*$. Then we know from Lemma 2.4 that the $S^{-1}R$-module $S^{-1}M$ satisfies strong $accr^*$ (for the proof of this assertion, we do not need the assumption that S is countable). Assume that S is countable. Let N be any submodule of M. Suppose that S is finite. Let $S = \{s_1, \ldots, s_t\}$. Let $s = \prod_{i=1}^{t} s_i$. Then it is clear that $s \in S$ and $Sat_S(N) = (N :_M s)$. Hence, we can assume that S is denumerable. Let $S = \{s_n | n \in \mathbb{N}\}$. Since M satisfies S-strong $accr^*$, the ascending sequence of submodules $(N :_M s_1) \subseteq (N :_M s_1 s_2) \subseteq (N :_M s_1 s_2 s_3) \subseteq \cdots$ is S-stationary. Therefore, there exist $s_i \in S$ and $k \in \mathbb{N}$ such that $s_i(N :_M s_1 \cdots s_n) \subseteq (N :_M s_1 \cdots s_k)$ for all $n \geq k$. Let $m \in Sat_S(N)$. Then $s_j m \in N$ for some $j \in \mathbb{N}$. Hence, $m \in (N :_M s_1 s_2 \cdots s_{k+j})$. Therefore, $s_i m \in s_i(N :_M s_1 s_2 \cdots s_{k+j}) \subseteq (N :_M s_1 \cdots s_k)$. This implies that $s_i s_1 \cdots s_k m \in N$. Let $s = s_i s_1 \cdots s_k$. Then $s \in S$ and $sm \in N$. This proves that $Sat_S(N) \subseteq (N :_M s)$ and it is clear that $(N :_M s) \subseteq Sat_S(N)$. Therefore, we get that $Sat_S(N) = (N :_M s)$.
$(ii) \Rightarrow (i)$ This follows from Lemma 2.6 (for this part of the proof, we do not need the assumption that S is countable). □

Let R be a ring. Recall from [3, page 321] that R is said to be *perfect* if every R-module has a projective cover. A pioneering work on perfect rings was done by Hyman Bass and there are several characterizations of perfect rings due to him [3, Theorem 28.4]. It was proved in [12, Proposition 1.1] that every R-module satisfies strong $accr^*$ if and only if R satisfies strong $accr^*$ and $dim\, R = 0$ which is equivalent to the statement that R is a perfect ring. Let S be a countable m.c. subset of R. As an application of [12, Proposition 1.1] and Theorem 2.7, in Theorem 2.8, we characterize rings R such that every module over R satisfies S-strong $accr^*$.

Theorem 2.8. *Let R be a ring. Let S be a countable m.c. subset of R. Suppose that for any module M over R and for any submodule N of M, there exists $s \in S$ (depending on N) such that $Sat_S(N) = (N :_M s)$. Then the following statements are equivalent:*

(i) Every module over R satisfies S-strong $accr^*$.
(ii) Every module over $S^{-1}R$ satisfies strong $accr^*$.
(iii) $S^{-1}R$ satisfies strong $accr^*$ and $dim(S^{-1}R) = 0$.
(iv) $S^{-1}R$ is a perfect ring.

Proof. $(i) \Rightarrow (ii)$ Let V be any module over $S^{-1}R$. Observe that V can be regarded as a module over R and hence by (i), V satisfies S-strong $accr^*$ regarded as a module over R. We know from Lemma 2.4 that the $S^{-1}R$-module $S^{-1}V = V$ satisfies strong $accr^*$. This shows that any $S^{-1}R$-module satisfies strong $accr^*$.
$(ii) \Rightarrow (iii)$ As any $S^{-1}R$-module satisfies strong $accr^*$, it follows that $S^{-1}R$ satisfies strong $accr^*$ and any $S^{-1}R$-module satisfies $accr^*$. Therefore, we obtain from $(ii) \Rightarrow (i)$ of [10, Proposition 2.4] that $dim(S^{-1}R) = 0$.
$(iii) \Rightarrow (iv)$ It follows from $(ii) \Rightarrow (iii)$ of [12, Proposition 1.1] that $S^{-1}R$ is a perfect ring.
$(iv) \Rightarrow (i)$ It follows from $(iii) \Rightarrow (i)$ of [12, Proposition 1.1] that any module over $S^{-1}R$ satisfies strong $accr^*$. Let M be a module over R. Now, the $S^{-1}R$-module $S^{-1}M$ satisfies strong $accr^*$. We are assuming that given any submodule N of M, there exists $s \in S$ (depending on N) such that $Sat_S(N) = (N :_M s)$. Therefore, we obtain from $(ii) \Rightarrow (i)$ of Theorem 2.7 that M satisfies S-strong $accr^*$. This proves that any module over R satisfies S-strong $accr^*$. □

We provide Example 2.9 to illustrate that $S^{-1}R$ is a perfect ring is not sufficient to imply that any module over R satisfies S-strong $accr^*$.

Example 2.9. Let us denote the set of all prime numbers by $\mathbb{P}$. Consider the $\mathbb{Z}$-module M given by $M = \mathbb{Z} + \sum_{p\in\mathbb{P}} \mathbb{Z}\frac{1}{p}$. Let $S = \mathbb{Z}\backslash\{0\}$. Then $S^{-1}\mathbb{Z}$ is a perfect ring and M does not satisfy S-strong $accr^*$.

Proof. Let $\mathbb{P} = \{p_1 = 2 < p_2 = 3 < p_3 < \cdots\}$. Note that $S^{-1}\mathbb{Z} = \mathbb{Q}$ is the field of rational numbers and so, $S^{-1}\mathbb{Z}$ is a perfect ring. It is not hard to verify that $(\mathbb{Z} :_M p_1 \cdots p_n) = \mathbb{Z} + \mathbb{Z}\frac{1}{p_1} + \cdots + \mathbb{Z}\frac{1}{p_n}$. Hence, the ascending sequence of submodules $(\mathbb{Z} :_M p_1) \subset (\mathbb{Z} :_M p_1p_2) \subset (\mathbb{Z} :_M p_1p_2p_3) \subset \cdots$ of M is strictly ascending. We claim that the above ascending sequence of submodules of M is not S-stationary. Suppose that the above ascending sequence of submodules of M is S-stationary. Then there exist $s \in S$ and $k \in \mathbb{N}$ such that $s(\mathbb{Z} :_M p_1 \cdots p_n) \subseteq (\mathbb{Z} :_M p_1 \cdots p_k)$ for all $n \geq k$. We can assume that $s > 0$. It is clear that $s \neq 1$. Note that there exist distinct $p_{i_1}, \ldots, p_{i_t} \in \mathbb{P}$ and positive integers $n_1, \ldots, n_t$ such that $s = \prod_{j=1}^{t} p_{i_j}^{n_j}$. Observe that $s(\mathbb{Z} :_M p_1p_2 \cdots p_{k+i_1+\cdots+i_t}) \subseteq \mathbb{Z} + \mathbb{Z}\frac{1}{p_1} + \cdots + \mathbb{Z}\frac{1}{p_k}$. This implies that $\frac{s}{p_{k+i_1+\cdots+i_t}} = \frac{m}{p_1\cdots p_k}$ for some $m \in \mathbb{Z}$. This is impossible since $p_{k+i_1+\cdots+i_t}$ does not divide $sp_1 \cdots p_k$ in $\mathbb{Z}$. This proves that the sequence of submodules $(\mathbb{Z} :_M p_1) \subset (\mathbb{Z} :_M p_1p_2) \subset (\mathbb{Z} :_M p_1p_2p_3) \subset \cdots$ is not S-stationary. This shows that M does not satisfy S-strong $accr^*$. □

Let M be a module over a ring R. Let $n \in \mathbb{N}$. Recall from [6] that M is said to *satisfy n-acc* if every ascending sequence of submodules of M, each of which is generated by n elements stabilizes. Recall from [6] that M is said to *satisfy pan-acc* if M satisfies n-acc for all $n \geq 1$. We say that R *satisfies n-acc* (respectively, *satisfies pan-acc*) if R regarded as a module over R satisfies n-acc (respectively, pan-acc). It is known that every module over a ring R satisfies pan-acc if and only if R is a perfect ring [9, Proposition 1.2].

Let S be a m.c. subset of a ring R. Let M be a module over R. Let $n \geq 1$. We say that M *satisfies S-n-acc* if any ascending sequence of submodules of M, each of which is generated by n elements is S-stationary. We say that M *satisfies S-pan-acc* if M satisfies S-n-acc for all $n \geq 1$. We say that R *satisfies S-n-acc* (respectively, *satisfies S-pan-acc*) if R regarded as a module over R satisfies S-n-acc (respectively, S-pan-acc).

We know from [6, Example pages 275–276] that there exist a domain D and a m.c. subset S of D such that D has pan-acc but $S^{-1}D$ does not have 1-acc. Let $n \geq 1$. The above mentioned example illustrates that if a module M over a ring R satisfies S- n-acc (S is a m.c. subset of R), then it need not imply that the $S^{-1}R$ -module $S^{-1}M$ has n-acc.

Lemma 2.10. *Let S be a m.c. subset of a ring R. Let $n \in \mathbb{N}$. Let M be a module over R. Suppose that for each submodule N of M generated by n elements, there exists $s \in S$ (depending on N) such that $Sat_S(N) = (N :_M s)$. If the $S^{-1}R$-module $S^{-1}M$ satisfies n-acc, then M satisfies S-n-acc.*

Proof. Let $N_1 \subseteq N_2 \subseteq N_3 \subseteq \cdots$ be an ascending sequence of submodules of M such that N_i is generated by n elements for each $i \in \mathbb{N}$. Observe that $S^{-1}N_1 \subseteq S^{-1}N_2 \subseteq S^{-1}N_3 \subseteq \cdots$ is an ascending sequence of n-generated submodules of $S^{-1}M$. Since $S^{-1}M$ satisfies n-acc by hypothesis, we obtain that there exists $k \in \mathbb{N}$ such that $S^{-1}N_n = S^{-1}N_k$ for all $n \geq k$. Hence, $Sat_S(N_n) = Sat_S(N_k)$ for all $n \geq k$. Moreover, by assumption,

$Sat_S(N_k) = (N_k :_M s)$ for some $s \in S$. Let $n \geq k$ and let $x \in N_n$. Then $x \in Sat_S(N_n) = Sat_S(N_k) = (N_k :_M s)$ and so, $sx \in N_k$. This shows that $sN_n \subseteq N_k$ for all $n \geq k$. This proves that any ascending sequence of submodules of M, each of which is n-generated is S-stationary. Therefore, we obtain that M satisfies S-n-acc. □

We provide Example 2.11 to illustrate that the hypothesis, for any submodule N of M generated by n elements, there exists $s \in S$ (depending on N) such that $Sat_S(N) = (N :_M s)$ in Lemma 2.10 cannot be omitted.

Example 2.11. Let $\mathbb{P}, M$ be as in Example 2.9. Let $S = \mathbb{Z} \backslash 2\mathbb{Z}$. Then the $S^{-1}\mathbb{Z}$-module $S^{-1}M$ satisfies pan-acc but M does not satisfy S-1-acc.

Proof. Now, $M = \mathbb{Z} + \sum_{p \in \mathbb{P}} \mathbb{Z}\frac{1}{p}$. It is not hard to verify that $M_{2\mathbb{Z}} = \mathbb{Z}_{2\mathbb{Z}} + \mathbb{Z}_{2\mathbb{Z}}\frac{1}{2}$. Observe that $M_{2\mathbb{Z}}$ is a Noetherian $\mathbb{Z}_{2\mathbb{Z}}$-module and so, $M_{2\mathbb{Z}}$ satisfies pan-acc. We next verify that M does not satisfy S-1-acc. Let $\mathbb{P} = \{p_1 = 2 < p_2 = 3 < p_3 < \cdots\}$. Let $n \geq 1$. It is convenient to denote $\frac{1}{p_1 \cdots p_n}$ by q_n and $\mathbb{Z}q_n$ by M_n. It is clear that $M_n \subset M$. Observe that $q_n = p_{n+1}q_{n+1}$ and so, $M_n \subseteq M_{n+1}$. From $\frac{1}{p_{n+1}} \in M_{n+1} \backslash M_n$, it follows that $M_n \subset M_{n+1}$. Hence, $M_! \subset M_2 \subset M_3 \subset \cdots$ is a strictly ascending sequence of 1-generated submodules of M. We claim that this sequence of cyclic submodules of M is not S-stationary. Suppose that this sequence of submodules of M is S-stationary. Then there exist $s \in S$ and $k \in \mathbb{N}$ such that $sM_n \subseteq M_k$ for all $n \geq k$. We can assume without loss of generality that $s > 0$. It is clear that $s \neq 1$. Observe that there exist $p_{i_1}, \ldots, p_{i_t} \in \mathbb{P} \backslash \{2\}$ and $n_1, \ldots, n_t \in \mathbb{N}$ such that $s = \prod_{j=1}^{t} p_{i_j}^{n_j}$. Note that $sM_{k+i_1+\cdots+i_t} \subseteq M_k$. This implies that $\frac{s}{p_1 p_2 \cdots p_{k+i_1+\cdots+i_t}} = \frac{y}{p_1 \cdots p_k}$ for some $y \in \mathbb{Z}$. This is impossible, since $p_{k+i_1+\cdots+i_t}$ does not divide $sp_1 \cdots p_k$ in $\mathbb{Z}$. Therefore, the sequence $M_1 \subset M_2 \subset M_3 \subset \cdots$ of 1-generated submodules of M is not S-stationary and so, M does not satisfy S-1-acc. □

Corollary 2.12. *Let S be a countable m.c. subset of a ring R. Suppose that for any module M over R, and any submodule N of M, there exists $s \in S$ (depending on N) such that $Sat_S(N) = (N :_M s)$. Consider the following statements.*
(i) Every module over R satisfies S-strong accr.*
(ii) $S^{-1}R$ is a perfect ring.
(iii) Every module over R satisfies S-pan-acc.
Then $(i) \Leftrightarrow (ii)$ and $(ii) \Rightarrow (iii)$.

Proof. $(i) \Leftrightarrow (ii)$ This is $(i) \Leftrightarrow (iv)$ of Theorem 2.8.
$(ii) \Rightarrow (iii)$ Let M be a module over R. Let $n \geq 1$. Since $S^{-1}R$ is a perfect ring, we obtain from [9, Proposition 1.2] that the $S^{-1}R$-module $S^{-1}M$ satisfies n-acc. Now, it follows from Lemma 2.10 that M satisfies S-n-acc. This is true for any $n \geq 1$. This proves that any R-module M satisfies S-pan-acc. □

It was shown in [12, Proposition 2.2] that if a domain R satisfies strong $accr^*$, then R satisfies 1-acc. In Example 2.13, we provide a domain R and a m.c. subset S of R such that R satisfies S-strong $accr^*$ but R does not satisfy strong $accr^*$.

Example 2.13. Let p be a prime number and let $F = \frac{Z}{p\mathbb{Z}}$. Let X be an indeterminate over F. For each $n \geq 1$, let $X^{\frac{1}{p^n}}$ denote the p^n-th root of X in an algebraic closure of $F(X)$.

Let $R = \cup_{n=1}^{\infty} F[X^{\frac{1}{p^n}}]$. Let $S = R\backslash\{0\}$. Then R satisfies S-strong $accr^*$ but R does not satisfy strong $accr^*$.

Proof. The domain R was considered by D.E. Dobbs in [5]. Let us denote the quotient field of R by K. Observe that $S^{-1}R = K$ is a Noetherian ring. Let I be any nonzero ideal of R. Then $S^{-1}I = K$ and so, $Sat_S(I) = K \cap R = R = (I :_R s)$ for any $s \in I\backslash\{0\}$. Hence, we obtain from [2, Proposition 2(f)] that R is S-Noetherian. Therefore, any increasing sequence of ideals of R is S-stationary and so, we get that R satisfies S-strong $accr^*$. Also, R satisfies S-pan-acc. Observe that $RX \subset RX^{\frac{1}{p}} \subset RX^{\frac{1}{p^2}} \subset \cdots$ is a strictly ascending sequence of principal ideals of R and so, R does not satisfy 1-acc. Hence, it follows from [12, Proposition 2.2] that R does not satisfy strong $accr^*$. □

3. Some more results on modules satisfying S-strong $accr^*$

Let R be a ring and let S be a m.c. subset of R. The aim of this section is to discuss some more results on modules M over R satisfying S-strong $accr^*$.

Remark 3.1. Let R be a ring and let S be a m.c. subset of R. Let M be a module over R. If M satisfies S-strong $accr^*$, then it is not hard to show that for any submodule N of M, both N and $\frac{M}{N}$ satisfy S-strong $accr^*$. We verify in Lemma 3.2 that if both N and $\frac{M}{N}$ satisfy S-strong $accr^*$, then M satisfies S-strong $accr^*$.

Lemma 3.2. *Let S be a m.c. subset of a ring R. Let M be a module over R and let N be a submodule of M. If both N and $\frac{M}{N}$ satisfy S-strong accr*, then M satisfies S-strong accr*.*

Proof. Let L be a submodule of M and let $\langle r_n \rangle$ be a sequence of elements of R. We verify that the ascending sequence $(L :_M r_1) \subseteq (L :_M r_1r_2) \subseteq (L :_M r_1r_2r_3) \subseteq \cdots$ of M is S-stationary. Since $\frac{M}{N}$ satisfies S-strong $accr^*$, there exist $s \in S$ and $k_1 \in \mathbb{N}$ such that for all $n \geq k_1$, $s(\frac{L+N}{N} :_{\frac{M}{N}} r_1 \cdots r_n) \subseteq (\frac{L+N}{N} :_{\frac{M}{N}} r_1 \cdots r_{k_1})$. This implies that $s(L + N :_M r_1 \cdots r_n) \subseteq (L + N :_M r_1 \cdots r_{k_1})$ for all $n \geq k_1$. As N satisfies S-strong $accr^*$, it follows that the ascending sequence of submodules $(N \cap L :_N r_{k_1+1}) \subseteq (N \cap L :_N r_{k_1+1}r_{k_1+2}) \subseteq (N \cap L :_N r_{k_1+1}r_{k_1+2}r_{k_1+3}) \subseteq \cdots$ of N is S-stationary. Hence, there exist $s' \in S$ and $k_2 \in \mathbb{N}$ such that for all $j \geq 1$, $s'(N \cap L :_N r_{k_1+1} \cdots r_{k_1+k_2+j}) \subseteq (N \cap L :_N r_{k_1+1} \cdots r_{k_1+k_2})$. We verify that $ss'(L :_M r_1r_2 \cdots r_n) \subseteq (L :_M r_1r_2 \cdots r_{k_1+k_2})$ for all $n \geq k_1 + k_2$. Let $n \geq k_1 + k_2$. Then $n = k_1 + k_2 + j$ for some $j \geq 0$. Let $m \in (L :_M r_1r_2 \cdots r_n)$. Now, $r_1r_2 \cdots r_n m \in L \subseteq L + N$. Hence, $sr_1 \cdots r_{k_1} m \in L + N$. This implies that $sr_1 \cdots r_{k_1} m = y + z$ for some $y \in L$ and $z \in N$. Therefore, $sr_1 \cdots r_{k_1} r_{k_1+1} \cdots r_n m = r_{k_1+1} \cdots r_n y + r_{k_1+1} \cdots r_n z$. Hence, $r_{k_1+1} \cdots r_n z \in N \cap L$. So, $s'r_{k_1+1} \cdots r_{k_1+k_2} z \in L \cap N$. Therefore, $ss'r_1r_2 \cdots r_{k_1+k_2} m = s'r_{k_1+1} \cdots r_{k_1+k_2} y + s'r_{k_1+1} \cdots r_{k_1+k_2} z \in L$. This proves that for all $n \geq k_1 + k_2$, $ss'(L :_M r_1 \cdots r_n) \subseteq (L :_M r_1 \cdots r_{k_1+k_2})$. Therefore, M satisfies S-strong $accr^*$. □

Let S be a m.c. subset of a ring R. As an application of Lemma 3.2, we verify in Remark 3.3 that if R satisfies S-strong $accr^*$, then M satisfies S-strong $accr^*$ for any f.g. R-module M.

Remark 3.3. Let S be a m.c. subset of a ring R. If $(0) \to M' \to M \to M'' \to (0)$ is a short exact sequence of R-modules, then using standard arguments, it follows from Remark 3.1 and Lemma 3.2 that M satisfies S-strong $accr^*$ if and only if both M' and M'' satisfy S-strong $accr^*$. If R satisfies S-strong $accr^*$, then for any $n \geq 1$, the free R-module R^n satisfies S-strong $accr^*$. If M is a f.g. module over a ring R, then M is a homomorphic image of a f.g. free R-module F. Thus if R satisfies S-strong $accr^*$, then M satisfies S-strong $accr^*$. □

Let R be an integral domain with $dim R > 0$. It was shown in [11, Result 12] that if a free R-module F satisfies $accr^*$, then F is f.g. As a consequence of this result and Remark 3.3, we prove in Proposition 3.4 that a free R-module F satisfies S-strong $accr^*$ if and only if F is f.g., where S is a m.c. subset of an integral domain R such that R satisfies S-strong $accr^*$ and $S^{-1}R$ is not a field.

Proposition 3.4. *Let R be an integral domain. Let S be a m.c. subset of R such that $S^{-1}R$ is not a field. Suppose that R satisfies S-strong $accr^*$. Let F be a free R-module. Then the following statements are equivalent:*

(i) F satisfies S-strong $accr^*$.
(ii) F satisfies S- $accr^*$.
(iii) F is finitely generated.

Proof. $(i) \Rightarrow (ii)$ This follows immediately from the fact that if a module M over a ring R satisfies S-strong $accr^*$, then M satisfies S-$accr^*$.
$(ii) \Rightarrow (iii)$ Let $\{e_\alpha\}_{\alpha \in \Lambda}$ be a basis of F as a free R-module. Then $S^{-1}F$ is a free $S^{-1}R$-module with basis $\{\frac{e_\alpha}{1}\}_{\alpha \in \Lambda}$. We are assuming that F satisfies S-$accr^*$. It can be shown as in Lemma 2.4 that the $S^{-1}R$-module $S^{-1}F$ satisfies $accr^*$. Since $S^{-1}R$ is not a field by assumption, we obtain from [11, Result 12] that Λ is a finite set. Therefore, it follows that F is finitely generated.
$(iii) \Rightarrow (i)$ By hypothesis, R satisfies S-strong $accr^*$. Since F is a f.g. module over R, we obtain from Remark 3.3 that F satisfies S-strong $accr^*$. □

We provide Example 3.5 to illustrate that the hypothesis F is a free module cannot be omitted in Proposition 3.4.

Example 3.5. Let p be a prime number and $R = \mathbb{Z}_{p\mathbb{Z}}$. It is well-known that R is a rank one discrete valuation domain with $\mathfrak{m} = pR$ as its unique maximal ideal. For each $n \in \mathbb{N}$, let M_n be the R-module given by $M_n = \frac{R}{\mathfrak{m}}$. Let $M = \bigoplus_{n \in \mathbb{N}} M_n$. Then M satisfies strong $accr^*$.

Proof. Since R is Noetherian, R satisfies S-strong $accr^*$ for any m.c. subset S of R. Let $S = 1 + \mathfrak{m}$. Then S is a m.c. subset of R and as $S \subseteq U(R)$, we obtain that $S^{-1}R = R$. As $\mathfrak{m}M$ is the zero submodule of M, M can be made into a module over $\frac{R}{\mathfrak{m}}$ by defining $(r + \mathfrak{m})x = rx$ for any $r + \mathfrak{m} \in \frac{R}{\mathfrak{m}}$ and for any $x \in M$. Observe that a nonempty subset N of M is an R-submodule of M if and only if N is an $\frac{R}{\mathfrak{m}}$-submodule of M. As M is a vector space over the field $\frac{R}{\mathfrak{m}}$, we know from Lemma 2.2 that M regarded as a module over $\frac{R}{\mathfrak{m}}$ satisfies strong $accr^*$ and so, M satisfies strong $accr^*$ as a module over R. Hence, M satisfies S-strong $accr^*$. But M is not a f.g. module over R.

Theorem 3.6. *Let S be a countable m.c. subset of a ring R. Then the following statements are equivalent:*

(i) *$R[X]$ satisfies S-strong $accr^*$.*

(ii) *$R[X]$ satisfies S-$accr^*$ and for any ideal A of $R[X]$, $Sat_S(A) = (A :_{R[X]} s)$ for some $s \in S$.*

(iii) *$R[X]$ is S-Noetherian.*

Proof. $(i) \Rightarrow (ii)$ As $R[X]$ satisfies S-strong $accr^*$, it is clear that $R[X]$ satisfies S-$accr^*$. Since S is a countable m.c. subset of R and $R[X]$ satisfies S-strong $accr^*$, it follows from $(i) \Rightarrow (ii)$ of Theorem 2.7 that if A is any ideal of $R[X]$, then $Sat_S(A) = (A :_{R[X]} s)$ for some $s \in S$.

$(ii) \Rightarrow (iii)$ As $R[X]$ satisfies S-$accr^*$, it follows as in Lemma 2.4 that $S^{-1}(R[X])$ satisfies $accr^*$. Observe that $S^{-1}(R[X]) = (S^{-1}R)[X]$ is the polynomial ring in one variable over $S^{-1}R$. Hence, we obtain from [8, Theorem 2] that $S^{-1}R$ is Noetherian. Therefore, it follows from Hilbert Basis Theorem [4, Theorem 7.5] that $(S^{-1}R)[X] = S^{-1}(R[X])$ is Noetherian. Thus $S^{-1}(R[X])$ is Noetherian and for any ideal A of $R[X]$, there exists $s \in S$ such that $Sat_S(A) = (A :_{R[X]} s)$. Hence, we obtain from [2, Proposition 2(f)] that $R[X]$ is S-Noetherian.

$(iii) \Rightarrow (i)$ Since $R[X]$ is S-Noetherian, any ascending sequence of ideals of $R[X]$ is S-stationary and so, $R[X]$ satisfies S-strong $accr^*$. □

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