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Skew-signings of positive weighted digraphs

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Abstract. An *arc-weighted digraph* is a pair (D, ω) where D is a digraph and ω is an *arc-weight function* that assigns to each arc uv of D a nonzero real number $\omega(uv)$. Given an arc-weighted digraph (D, ω) with vertices $v_1, \ldots, v_n$, the *weighted adjacency matrix* of (D, ω) is defined as the $n \times n$ matrix $A(D, \omega) = [a_{ij}]$ where $a_{ij} = \omega(v_i v_j)$ if $v_i v_j$ is an arc of D, and 0 otherwise. Let (D, ω) be a positive arc-weighted digraph and assume that D is loopless and symmetric. A *skew-signing* of (D, ω) is an arc-weight function ω' such that $\omega'(uv) = \pm\omega(uv)$ and $\omega'(uv)\omega'(vu) < 0$ for every arc uv of D. In this paper, we give necessary and sufficient conditions under which the characteristic polynomial of $A(D, \omega')$ is the same for all skew-signings ω' of (D, ω). Our main theorem generalizes a result of Cavers et al. (2012) about skew-adjacency matrices of graphs.

Keywords: Arc-weighted digraphs; Skew-signing of a digraph; Weighted adjacency matrix

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1. INTRODUCTION

A *directed graph* or, more simply, a *digraph* D is a pair $D = (V, E)$ where V is a set of *vertices* and E is a set of ordered pairs of vertices called *arcs*. For $u, v \in V$, the arc $a = (u, v)$

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of D is denoted by uv. An arc of the form uu is called a *loop* of D. A *loopless digraph* is one containing no loops. A *symmetric* digraph is a digraph such that if uv is an arc then vu is also an arc. Given a symmetric digraph $D = (V, E)$ and a subdigraph $H = (W, F)$ of D, we denote by H^* the subdigraph of D whose vertex set is W and arc set is $\{vu : uv \in F\}$.

Let G be a simple undirected and finite graph. An *orientation* of G is an assignment of a direction to each edge of G so that we obtain a directed graph $\overrightarrow{G}$. Let $\overrightarrow{G}$ be an orientation of G. With respect to a labeling $v_1, \ldots, v_n$ of the vertices of G, the *skew-adjacency* matrix of $\overrightarrow{G}$ is the $n \times n$ real skew-symmetric matrix $S(\overrightarrow{G}) = [s_{ij}]$, where $s_{ij} = 1$ and $s_{ji} = -1$ if $v_i v_j$ is an arc of $\overrightarrow{G}$, otherwise $s_{ij} = s_{ji} = 0$. The *skew-characteristic polynomial* of $\overrightarrow{G}$ is defined as the characteristic polynomial of $S(\overrightarrow{G})$. This definition is correct because skew-adjacency matrices of $\overrightarrow{G}$ with respect to different labelings are permutationally similar and so have the same characteristic polynomial.

There are several recent works about skew-characteristic polynomials of oriented graphs, one can see for example [1,3–6,10]. An open problem is to find the number of possible orientations with distinct skew-characteristic polynomials of a given graph G. In particular it is of interest to know whether all orientations of a graph G can have the same skew-characteristic polynomial. The following theorem, obtained by Cavers et al. [3] gives an answer to this question.

Theorem 1.1. *The orientations of a graph G all have the same skew-characteristic polynomial if and only if G has no cycles of even length.*

A similar result to this theorem was obtained by Liu and Zhang [7]. They proved that all orientations of a graph G have the same permanental polynomial if and only if G has no cycles of even length.

In this work, we will extend Theorem 1.1 to positive weighted loopless and symmetric digraphs (which we abbreviate to *pwls-digraphs*). *An arc-weighted digraph* or more simply a *weighted digraph* is a pair (D, ω) where D is a digraph and ω is a *arc-weight function* that assigns to each arc uv of D a nonzero real number $\omega(uv)$, called *the weight* of the arc uv. Let (D, ω) be a weighted digraph with vertices $v_1, \ldots, v_n$. The weighted adjacency matrix of (D, ω) is defined as the $n \times n$ matrix $A(D, \omega) = [a_{ij}]$ where $a_{ij} = \omega(v_i v_j)$, if $v_i v_j$ is an arc of D and 0 otherwise. Let (D, ω) be a pwls-digraph. A skew-signing of (D, ω) is an arc-weight function ω' such that $\omega'(uv) = \pm\omega'(uv)$ and $\omega'(uv)\omega'(vu) < 0$ for every arc uv of D.

Our aim is to characterize the pwls-digraphs (D, ω) for which the characteristic polynomial of $A(D, \omega')$ is the same for all skew-signings ω' of (D, ω). This characterization involves directed cycles in D. Recall that a *directed cycle* of length $t > 0$ is a digraph with vertex set $\{v_1, v_2, \ldots, v_t\}$ and arcs $v_1 v_2, \ldots, v_{t-1} v_t, v_t v_1$. Throughout this paper, we use the term “cycle” to refer to a “directed cycle” in a digraph. A cycle of length $t = 2$ is called a *digon*. A cycle is *odd* (resp. *even*) if its length is odd (resp. even). Our main result is the following theorem.

Theorem 1.2. *Let (D, ω) be a pwls-digraph. Then the following statements are equivalent:*

(i) *The characteristic polynomial of (D, ω') is the same for all skew-signings ω' of (D, ω).*
(ii) *D has no even cycles of length more than 2 and $A(D, \omega) = \Delta^{-1} S \Delta$ where S is a nonnegative symmetric matrix with zero diagonal and Δ is a diagonal matrix with positive diagonal entries.*

Note that a graph G can be identified with the pwls-digraph obtained from G by replacing each edge joining two vertices u and v by the arcs uv and vu, both of them have weight 1. Moreover, every orientation of G can be identified with a skew-signing of this weighted digraph. Then, our main result is a generalization of Theorem 1.1.

2. CYCLE-SYMMETRIC DIGRAPHS

There is a natural correspondence between real matrices and weighted digraphs. Indeed, every $n \times n$ real matrix $M = [m_{ij}]$ is the weighted adjacency matrix of a unique weighted digraph (D_M, ω_M) with vertex set $\{1, \dots, n\}$. This digraph, called the *weighted digraph associated* to M, is defined as follows: ij is an arc of D_M iff $m_{ij} \neq 0$, and the weight of an arc ij is $\omega_M(ij) = m_{ij}$.

We start with some formulas involving the characteristic polynomial of a matrix and its weighted associated digraph. For this, we need some notations and definitions. Let D be a digraph. A *linear subdigraph* L of D is a vertex disjoint union of some cycles in D. A linear subdigraph L of D is called *even linear* if L contains no odd cycles. Let $\overrightarrow{\mathcal{L}}_k(D)$ (resp. $\overrightarrow{\mathcal{L}}^e_k(D)$) denote the set of all linear (resp. even linear) subdigraphs of D that cover precisely k vertices of D. We usually write this as $\overrightarrow{\mathcal{L}}_k$ (resp. $\overrightarrow{\mathcal{L}}^e_k$) when no ambiguity can arise.

Let A be a real matrix and let (D, ω) be the weighted digraph associated to A. We denote by $p_A(x) = \det(xI - A) = x^n + a_1x^{n-1} + \dots + a_{n-1}x + a_n$ the characteristic polynomial of A. *The Coates determinant formula* (see [2] p. 65) can be stated as follows:

$$a_k = \sum_{\overrightarrow{L} \in \overrightarrow{\mathcal{L}}_k} (-1)^{\left|\overrightarrow{L}\right|} \omega(\overrightarrow{L}) \tag{1}$$

where $\left|\overrightarrow{L}\right|$ denotes the number of cycles in $\overrightarrow{L}$ and $\omega(\overrightarrow{L})$ is the product of all the weights of the arcs of $\overrightarrow{L}$.

In particular

$$\det(A) = (-1)^n a_n = (-1)^n \sum_{\overrightarrow{L} \in \overrightarrow{\mathcal{L}}_n} (-1)^{\left|\overrightarrow{L}\right|} \omega(\overrightarrow{L}). \tag{2}$$

We consider now the case where A is skew-symmetric. Let $\overrightarrow{C}$ be a cycle of length k of D. Then

$$\begin{aligned}\omega(\overrightarrow{C}) &= a_{i_1i_2}a_{i_2i_3}\dots a_{i_{k-1}i_k}a_{i_ki_1} = (-1)^k a_{i_1i_k}a_{i_ki_{k-1}}\dots a_{i_3i_2}a_{i_2i_1}\\ &= (-1)^k\omega(\overrightarrow{C^*})\\ &= \begin{cases} -\omega(\overrightarrow{C^*}) \text{ if } k \text{ is odd}\\ \omega(\overrightarrow{C^*}) \text{ if } k \text{ is even.}\end{cases}\end{aligned}$$

Let $\overrightarrow{L} \in \overrightarrow{\mathcal{L}_k} \setminus \overrightarrow{\mathcal{L}^e_k}$. By definition of $\overrightarrow{\mathcal{L}_k}$ and $\overrightarrow{\mathcal{L}^e_k}$, the linear subdigraph $\overrightarrow{L}$ contains an odd cycle $\overrightarrow{C}$ among its components. Let $\overrightarrow{L'}$ the linear subdigraph obtained from $\overrightarrow{L}$ by replacing the cycle $\overrightarrow{C}$ by $\overrightarrow{C^*}$. Since $\overrightarrow{C}$ is odd, and then $\omega(\overrightarrow{C}) = -\omega(\overrightarrow{C^*})$, $\omega(\overrightarrow{L}) = -\omega(\overrightarrow{L'})$. Thus, linear subdigraphs of $\overrightarrow{\mathcal{L}_k} \setminus \overrightarrow{\mathcal{L}^e_k}$ contribute 0 to a_k.

It follows that

$$a_k = \sum_{\overrightarrow{L} \in \overrightarrow{\mathcal{L}}_k^e} (-1)^{\left|\overrightarrow{L}\right|} \omega(\overrightarrow{L}).$$

If k is odd, then $\overrightarrow{\mathcal{L}}_k^e$ is empty, and hence

$$a_k = \begin{cases} 0 \text{ if } k \text{ is odd} \\ \sum_{\overrightarrow{L} \in \overrightarrow{\mathcal{L}}_k^e} (-1)^{\left|\overrightarrow{L}\right|} \omega(\overrightarrow{L}) \text{ if } k \text{ is even.} \end{cases} \tag{3}$$

We introduce now a special class of weighted symmetric digraphs called cycle-symmetric digraphs. The characterization of these digraphs will be used in the proof of our main theorem.

Let ω be a positive arc-weight function of D and let q be a positive integer. We say that (D, ω) is $(\leq q)$-*cycle-symmetric* if $\omega(\overrightarrow{C}) = \omega(\overrightarrow{C}^*)$ for every cycle $\overrightarrow{C}$ of D of length at most q. If $q = n$ then (D, ω) is said to be *cycle-symmetric*.

We borrowed the terminology "cycle-symmetric" from Shih and Weng [11]. Following this paper, an $n \times n$ real matrix $\left[a_{ij}\right]$ is called *cycle-symmetric* if the following two conditions hold:

(C1) for $i \neq j \in \{1, \ldots, n\}$, $a_{ij}a_{ji} > 0$ or $a_{ij} = a_{ji} = 0$;
(C2) For any sequence of distinct integers $i_1, \ldots, i_k$ from the set $\{1, \ldots, n\}$, we have

$$a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_{k-1} i_k} a_{i_k i_1} = a_{i_1 i_k} a_{i_k i_{k-1}} \cdots a_{i_3 i_2} a_{i_2 i_1}.$$

Remark 1. Obviously, a pwls-digraph (D, ω) is always (≤ 2)-cycle-symmetric. Moreover, a pwls-digraph is cycle-symmetric if and only if its weighted adjacency matrix is cycle-symmetric.

The following theorem gives a characterization of cycle-symmetric matrices. For the proof, one can see [8,9,11].

Theorem 2.1. *An $n \times n$ real matrix A is cycle-symmetric if and only if there exists an invertible diagonal matrix D such that $D^{-1}AD$ is symmetric.*

It easy to see that if A is a nonnegative matrix, then the diagonal entries of D can be chosen positive. So by using Remark 1, we obtain the following corollary.

Corollary 2.2. *Let (D, ω) be a pwls-digraph. Then, the following statements are equivalent:*

(i) *(D, ω) is cycle-symmetric.*
(ii) *$A(D, \omega) = \Delta^{-1} S \Delta$ where S is a nonnegative symmetric matrix with zero diagonal and Δ is a diagonal matrix with positive diagonal entries.*

3. Skew-signings of cycle-symmetric digraphs

In this section, we will prove the following proposition, which is a special case of our main theorem for cycle-symmetric pwsl-digraphs.

Proposition 3.1. *Let (D, ω) be a cycle-symmetric pwls-digraph. Then, the following statements are equivalent:*

(i) *The characteristic polynomial of (D, ω') is the same for all skew-signings ω' of (D, ω);*
(ii) *D contains no even cycles of length greater than 3.*

We start with some preparatory results about skew-signings of pwls-digraphs. Let (D, ω) be an arbitrary pwls-digraph and let ω' be a skew-signing of (D, ω). We consider the two arc-weight functions defined as follows: $\overline{\omega}(uv) = \sqrt{\omega(uv)\omega(vu)}$ and $\widehat{\omega'}(uv) = \frac{\omega'(uv)}{\omega(uv)}\sqrt{\omega(uv)\omega(vu)}$ for every arc uv of D. It is easy to check the following properties:

P1 The weighted adjacency matrix of $(D, \overline{\omega})$ is symmetric.
P2 $\widehat{\omega'}$ is a skew-signing of $(D, \overline{\omega})$ and the weighted adjacency matrix of $(D, \widehat{\omega'})$ is skew-symmetric.
P3 Let q be a positive integer such that $3 \leq q \leq n$. If (D, ω) is a $(\leq q)$-cycle-symmetric digraph, then for every cycle $\overrightarrow{C}$ of D with length at most q we have $\omega(\overrightarrow{C}) = \overline{\omega}(\overrightarrow{C})$ and $\omega'(\overrightarrow{C}) = \widehat{\omega'}(\overrightarrow{C})$.

We denote by $p_{(D,\omega)}(x) := x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$ the characteristic polynomial of (D, ω). The characteristic polynomials of (D, ω') and $(D, \widehat{\omega'})$ are respectively denoted by $p_{(D,\omega')}(x) := x^n + b_1x^{n-1} + \cdots + b_{n-1}x + b_n$ and $p_{(D,\widehat{\omega'})}(x) := x^n + c_1x^{n-1} + \cdots + c_{n-1}x + c_n$.
From Formula (1), we have $b_1 = 0$ and $b_2 = -a_2$. In particular, b_1 and b_2 are independent of ω'.

Lemma 3.2. *Let q be a positive integer such that $3 \leq q \leq n$. If (D, ω) is $(\leq q)$-cycle-symmetric, then:*

$$b_k = \begin{cases} 0 & \text{if } k \text{ is odd} \\ \displaystyle\sum_{\overrightarrow{L} \in \overrightarrow{\mathcal{L}}_k^e} (-1)^{|\overrightarrow{L}|} \omega'(\overrightarrow{L}) & \text{if } k \text{ is even} \end{cases}$$

for $k = 1, \ldots, q$.

Proof. Let $k \in \{1, \ldots, q\}$. From Property **P3**, we have $\omega'(\overrightarrow{L}) = \widehat{\omega'}(\overrightarrow{L})$ for every $\overrightarrow{L} \in \overrightarrow{\mathcal{L}}_k$. By using Formula (1), it follows that $b_k = c_k$. Moreover, from Property **P2**, $A(D, \widehat{\omega'})$ is a skew-symmetric matrix. Then by Formula (3) we have:

$$c_k = \begin{cases} 0 & \text{if } k \text{ is odd} \\ \displaystyle\sum_{\overrightarrow{L} \in \overrightarrow{\mathcal{L}}_k^e} (-1)^{|\overrightarrow{L}|} \widehat{\omega'}(\overrightarrow{L}) & \text{if } k \text{ is even.} \end{cases}$$

Now, by applying again **P3**, we obtain

$$b_k = c_k = \begin{cases} 0 & \text{if } k \text{ is odd} \\ \displaystyle\sum_{\overrightarrow{L} \in \overrightarrow{\mathcal{L}}_k^e} (-1)^{|\overrightarrow{L}|} \omega'(\overrightarrow{L}) & \text{if } k \text{ is even.} \end{cases} \quad \square$$

We denote by $\overrightarrow{\mathcal{C}}_k$ the set of all cycles of length k in D. For a skew-signing ω', this set can be partitioned into two subsets: $\overrightarrow{\mathcal{C}}^+_{k,\omega'}$ and $\overrightarrow{\mathcal{C}}^-_{k,\omega'}$ where $\overrightarrow{\mathcal{C}}^+_{k,\omega'}$ (resp. $\overrightarrow{\mathcal{C}}^-_{k,\omega'}$) is the set of cycles $\overrightarrow{C}$ with length k such that $\omega'(\overrightarrow{C}) > 0$ (resp. $\omega'(\overrightarrow{C}) < 0$). In the case when k is even, we denote by $\overrightarrow{\mathcal{D}}_k$ the set of all collections $\overrightarrow{L}$ of vertex disjoint digons that cover precisely k vertices in D.

Corollary 3.3. *Assume that (D, ω) is $(\leq q-1)$-cycle-symmetric for some $q \in \{4, \dots, n+1\}$ and contains no even cycles of length $k \in \{3, \dots, q-1\}$, then*

$$b_k = \begin{cases} 0 & \text{if } k \text{ is odd} \\ \sum_{\overrightarrow{L} \in \overrightarrow{\mathcal{D}}_k} \omega(\overrightarrow{L}) & \text{if } k \text{ is even} \end{cases}$$

for $k = 1, \dots, q-1$ and if $q \leq n$, then

$$b_q = \begin{cases} -\sum_{\overrightarrow{C} \in \overrightarrow{\mathcal{C}}^+_{q,\omega'}} \omega(\overrightarrow{C}) + \sum_{\overrightarrow{C} \in \overrightarrow{\mathcal{C}}^-_{q,\omega'}} \omega(\overrightarrow{C}) & \text{if } q \text{ is odd} \\ -\sum_{\overrightarrow{C} \in \overrightarrow{\mathcal{C}}^+_{q,\omega'}} \omega(\overrightarrow{C}) + \sum_{\overrightarrow{C} \in \overrightarrow{\mathcal{C}}^-_{q,\omega'}} \omega(\overrightarrow{C}) + \sum_{\overrightarrow{L} \in \overrightarrow{\mathcal{D}}_q} \omega(\overrightarrow{L}) & \text{if } q \text{ is even.} \end{cases}$$

Proof. The first equality follows from Lemma 3.2.

From Formula (1), we have

$$\begin{aligned} b_q &= \sum_{\overrightarrow{L} \in \overrightarrow{\mathcal{L}}_q} (-1)^{|\overrightarrow{L}|} \omega'(\overrightarrow{L}) \\ &= \sum_{\overrightarrow{L} \in \overrightarrow{\mathcal{L}}_q \smallsetminus (\overrightarrow{\mathcal{L}}^e_q \cup \overrightarrow{\mathcal{C}}_q)} (-1)^{|\overrightarrow{L}|} \omega'(\overrightarrow{L}) + \sum_{\overrightarrow{L} \in \overrightarrow{\mathcal{L}}^e_q \setminus \overrightarrow{\mathcal{C}}_q} (-1)^{|\overrightarrow{L}|} \omega'(\overrightarrow{L}) - \sum_{\overrightarrow{C} \in \overrightarrow{\mathcal{C}}_q} \omega'(\overrightarrow{C}). \end{aligned}$$

By definition of $\overrightarrow{\mathcal{C}}^+_{q,\omega'}$ and $\overrightarrow{\mathcal{C}}^-_{q,\omega'}$, we have

$$\sum_{\overrightarrow{C} \in \overrightarrow{\mathcal{C}}_q} \omega'(\overrightarrow{C}) = \sum_{\overrightarrow{C} \in \overrightarrow{\mathcal{C}}^+_{q,\omega'}} \omega(\overrightarrow{C}) - \sum_{\overrightarrow{C} \in \overrightarrow{\mathcal{C}}^-_{q,\omega'}} \omega(\overrightarrow{C}).$$

Consider now $\overrightarrow{L} \in \overrightarrow{\mathcal{L}}_q \smallsetminus (\overrightarrow{\mathcal{L}}^e_q \cup \overrightarrow{\mathcal{C}}_q)$. By definition of $\overrightarrow{\mathcal{L}}_q$ and $\overrightarrow{\mathcal{L}}^e_q$, the linear subdigraph $\overrightarrow{L}$ contains an odd cycle $\overrightarrow{C}$ among its components. Let $\overrightarrow{L}$ the linear subdigraph obtained from $\overrightarrow{L}$ by replacing the cycle $\overrightarrow{C}$ by $\overrightarrow{C^*}$. Since $\overrightarrow{C}$ is odd and $\omega(\overrightarrow{C}) = \omega(\overrightarrow{C^*})$, $\omega'(\overrightarrow{L}) = -\omega'(\overrightarrow{L'})$. Thus, linear subdigraphs of $\overrightarrow{\mathcal{L}}_q \smallsetminus (\overrightarrow{\mathcal{L}}^e_q \cup \overrightarrow{\mathcal{C}}_q)$ contribute 0 to b_q.

Now, according to the parity of q,we will distinguish two cases:

Case 1: If q is odd, then $\overrightarrow{\mathcal{L}}^e_q = \emptyset$ and hence

$b_q = -\sum_{\overrightarrow{C} \in \overrightarrow{\mathcal{C}}^+_{q,\omega'}} \omega(\overrightarrow{C}) + \sum_{\overrightarrow{C} \in \overrightarrow{\mathcal{C}}^-_{q,\omega'}} \omega(\overrightarrow{C})$.

Case 2: If q is even, then by hypothesis $\overrightarrow{\mathcal{L}}^e_q = \overrightarrow{\mathcal{D}}_q$ and hence

$b_q = \sum_{\overrightarrow{L} \in \overrightarrow{\mathcal{D}}_q} (-1)^{|\overrightarrow{L}|} \omega'(\overrightarrow{L}) - \sum_{\overrightarrow{C} \in \overrightarrow{\mathcal{C}}^+_{q,\omega'}} \omega(\overrightarrow{C}) + \sum_{\overrightarrow{C} \in \overrightarrow{\mathcal{C}}^-_{q,\omega'}} \omega(\overrightarrow{C})$.

This completes the proof of the second equality. □

The implication $(ii) \Rightarrow (i)$ of Proposition 3.1 is deduced form Corollary 3.3 for $q = n+1$.

Before proving the implication $(i) \Rightarrow (ii)$, we introduce some notations and establish an intermediate result. Let (D, ω) be an arbitrary pwsl-digraph and consider an arbitrary cycle of D of length $q \geqslant 3$ whose vertices are $v_1, \dots, v_q$ and whose arcs are $e_1 := v_1v_2, \dots, e_{q-1} := v_{q-1}v_q$, $e_q := v_qv_1$. Let ω' be a skew-signing of (D, ω). Let $h \in \{1, \dots, q\}$, we denote by $\eta^+_{\omega'}(e_1, \dots, e_h)$ the sum of the weights of cycles $\overrightarrow{C}$ of length q in D such that $\omega'(\overrightarrow{C}) > 0$ and contain arcs $e_1, \dots, e_h$. Define $\eta^-_{\omega'}(e_1, \dots, e_h)$ analogously. For $r < h$, we denote by $\eta^+_{\omega'}(e_1, \dots, e_r, \overline{e_{r+1}}, \dots, \overline{e_h})$ the sum of the weights of cycles $\overrightarrow{C}$ of length q in D such that $\omega'(\overrightarrow{C}) > 0$ and contain arcs $e_1, \dots, e_r$ but not arcs $e_{r+1}, \dots, e_h$,. Define $\eta^-_{\omega'}(e_1, \dots, e_r, \overline{e_{r+1}}, \dots, \overline{e_h})$ analogously.

Lemma 3.4. *There exists a skew-signing ω'_0 of (D, ω) such that $\eta^+_{\omega'_0}(e_1) \neq \eta^-_{\omega'_0}(e_1)$.*

Proof. Assume the contrary. We claim that for each $t \in \{1, \dots, q\}$ and for all skew-signings ω' of (D, ω), $\eta^+_{\omega'}(e_1, \dots, e_t) = \eta^-_{\omega'}(e_1, \dots, e_t)$. For this, we proceed by induction on t. The case $t = 1$ is assumed. Let $t \in \{1, \dots, q-1\}$ and suppose that the claim is true for t. Then

$$\begin{cases} \eta^+_{\omega'}(e_1, \dots, e_t) = \eta^+_{\omega'}(e_1, e_2, \dots, e_t, e_{t+1}) + \eta^+_{\omega'}(e_1, e_2, \dots, e_t, \overline{e_{t+1}}) \\ \eta^-_{\omega'}(e_1, \dots, e_t) = \eta^-_{\omega'}(e_1, e_2, \dots, e_t, e_{t+1}) + \eta^-_{\omega'}(e_1, e_2, \dots, e_t, \overline{e_{t+1}}). \end{cases}$$

Consider now the skew-signing ω'' that coincides with ω' outside $\{e_{t+1}, e^*_{t+1}\}$ and such that $\omega''(e) = -\omega'(e)$ for $e \in \{e_{t+1}, e^*_{t+1}\}$.

Then, we have

$$\begin{cases} \eta^+_{\omega''}(e_1, \dots, e_t) = \eta^-_{\omega'}(e_1, e_2, \dots, e_t, e_{t+1}) + \eta^+_{\omega'}(e_1, e_2, \dots, e_t, \overline{e_{t+1}}) \\ \eta^-_{\omega''}(e_1, \dots, e_t) = \eta^+_{\omega'}(e_1, e_2, \dots, e_t, e_{t+1}) + \eta^-_{\omega'}(e_1, e_2, \dots, e_t, \overline{e_{t+1}}). \end{cases}$$

But by induction hypothesis, we have $\eta^+_{\omega''}(e_1, \dots, e_t) = \eta^-_{\omega''}(e_1, \dots, e_t)$ and $\eta^+_{\omega'}(e_1, \dots, e_t) = \eta^-_{\omega'}(e_1, \dots, e_t)$.

Then

$$\begin{aligned} \eta^+_{\omega'}(e_1, \dots, e_t, \overline{e_{t+1}}) - \eta^-_{\omega'}(e_1, \dots, e_t, \overline{e_{t+1}}) &= \eta^-_{\omega'}(e_1, \dots, e_{t+1}) - \eta^+_{\omega'}(e_1, \dots, e_{t+1}) \\ &= \eta^+_{\omega'}(e_1, \dots, e_{t+1}) - \eta^-_{\omega'}(e_1, \dots, e_{t+1}). \end{aligned}$$

Thus $\eta^+_{\omega'}(e_1, \dots, e_t, e_{t+1}) = \eta^-_{\omega'}(e_1, e_2, \dots, e_t, e_{t+1})$.

This completes the induction proof. For $t = q$ we have, $\eta^+_{\omega'}(e_1, \dots, e_q) = \eta^-_{\omega'}(e_1, \dots, e_q)$.

Now, choose a skew-signing ω' of (D, ω) such that $\omega'(e_1) = \omega(e_1), \dots, \omega'(e_q) = \omega(e_q)$. Then, we have $\eta^+_{\omega'}(e_1, \dots, e_q) = \prod_{i=1}^{q} \omega(e_i)$ and $\eta^-_{\omega'}(e_1, \dots, e_q) = 0$, a contradiction. It follows that there exists a skew-signing ω'_0 such that $\eta^+_{\omega'_0}(e_1) \neq \eta^-_{\omega'_0}(e_1)$. $\square$

The proof of $(i) \Rightarrow (ii)$ in Proposition 3.1 follows from the following more general result.

Lemma 3.5. *Let (D, ω) be an $(\leq l)$-cycle-symmetric pwsl-digraph where $l \geqslant 3$. If the characteristic polynomial of (D, ω') is the same for all skew-signings ω' of (D, ω), then every cycle of length at most l is an odd cycle or a digon.*

Proof. Assume for contradiction that D contains an even cycle of length $q \in \{4, \dots, l\}$ and choose such a cycle with q as small as possible. We will use the notations of the previous lemma. Let ω'' be the skew-signing of (D,ω) that coincides with ω_0' outside $\{e_1, e_1^*\}$ and such that $\omega''(e) = -\omega_0'(e)$ for $e \in \{e_1, e_1^*\}$. The characteristic polynomials of (D,ω_0') and (D,ω'') are respectively denoted by $p_{(D,\omega_0')}(x) := x^n + b_1x^{n-1} + \cdots + b_{n-1}x + b_n$ and $p_{(D,\omega'')}(x) := x^n + c_1x^{n-1} + \cdots + c_{n-1}x + c_n$.

By the choice of q and from the second equality of Corollary 3.3, we have $b_q - c_q = -\sum_{\overrightarrow{C} \in \overrightarrow{\mathcal{C}}^+_{q,\omega_0'}} \omega(\overrightarrow{C}) + \sum_{\overrightarrow{C} \in \overrightarrow{\mathcal{C}}^-_{q,\omega_0'}} \omega(\overrightarrow{C}) + \sum_{\overrightarrow{C} \in \overrightarrow{\mathcal{C}}^+_{q,\omega''}} \omega(\overrightarrow{C}) - \sum_{\overrightarrow{C} \in \overrightarrow{\mathcal{C}}^-_{q,\omega''}} \omega(\overrightarrow{C})$.

Every cycle $\overrightarrow{C}$ of length q that contains neither e_1 nor e_1^* contributes 0 to $b_q - c_q$. It follows that:

$$\begin{aligned} b_q - c_q &= -\eta^+_{\omega_0'}(e_1) - \eta^+_{\omega_0'}(e_1^*) + \eta^-_{\omega_0'}(e_1) + \eta^-_{\omega_0'}(e_1^*) \\ &\quad + \eta^+_{\omega''}(e_1) + \eta^+_{\omega''}(e_1^*) - \eta^-_{\omega''}(e_1) - \eta^-_{\omega''}(e_1^*). \end{aligned}$$

By construction of ω'', we have $\eta^+_{\omega''}(e_1) = \eta^-_{\omega_0'}(e_1)$, $\eta^-_{\omega''}(e_1) = \eta^+_{\omega_0'}(e_1)$, $\eta^+_{\omega''}(e_1^*) = \eta^-_{\omega_0'}(e_1^*)$, $\eta^-_{\omega''}(e_1^*) = \eta^+_{\omega_0'}(e_1^*)$.

Then $b_q - c_q = -2(\eta^+_{\omega_0'}(e_1) + \eta^+_{\omega_0'}(e_1^*)) + 2(\eta^-_{\omega_0'}(e_1) + \eta^-_{\omega_0'}(e_1^*))$

As (D,ω) is $(\leq l)$-cycle-symmetric, we have $\eta^+_{\omega_0'}(e_1) = \eta^+_{\omega_0'}(e_1^*)$, $\eta^-_{\omega_0'}(e_1^*) = \eta^-_{\omega_0'}(e_1)$ and then $b_q - c_q = -4(\eta^+_{\omega_0'}(e_1) - \eta^-_{\omega_0'}(e_1)) \neq 0$, a contradiction. □

4. Proof of the main theorem

The implication $(ii) \Longrightarrow (i)$ follows easily from Corollary 2.2 and Proposition 3.1. To prove $(i) \Longrightarrow (ii)$ it suffices to use Proposition 3.1 and the next lemma.

Lemma 4.1. *Let (D,ω) be a pwls-digraph. If the characteristic polynomial of (D,ω') is the same for all skew-signings ω' of (D,ω), then (D,ω) is cycle-symmetric.*

Proof. Assume for a contradiction that (D,ω) is not cycle-symmetric and let $\overrightarrow{C}_0$ be a shortest cycle of D such that $\omega(\overrightarrow{C}_0) \neq \omega(\overrightarrow{C}^*_0)$. We denote by $v_1,\ldots,v_q$ the vertices of $\overrightarrow{C}_0$ and $e_1 := v_1v_2, \ldots, e_{q-1} := v_{q-1}v_q$, $e_q := v_qv_1$ its arcs. Let $h \in \{1,\ldots,q\}$ and $r \in \{1,\ldots,h\}$. For every skew-signing ω' of (D,ω),we set:

$$\begin{aligned} N^+_{\omega'}(e_1,\ldots,e_r,\overline{e_{r+1}},\ldots,\overline{e_h}) &= \eta^+_{\omega'}(e_1,\ldots,e_r,\overline{e_{r+1}},\ldots,\overline{e_h}) \\ &\quad + \eta^+_{\omega'}(e_1^*,\ldots,e_r^*,\overline{e_{r+1}^*},\ldots,\overline{e_h^*}) \\ N^-_{\omega'}(e_1,\ldots,e_r,\overline{e_{r+1}},\ldots,\overline{e_h}) &= \eta^-_{\omega'}(e_1,\ldots,e_r,\overline{e_{r+1}},\ldots,\overline{e_h}) \\ &\quad + \eta^-_{\omega'}(e_1^*,\ldots,e_r^*,\overline{e_{r+1}^*},\ldots,\overline{e_h^*}). \end{aligned}$$

Step 1 There exists a skew-signing ω_0' of (D,ω) such that $N^+_{\omega_0'}(e_1) \neq N^-_{\omega_0'}(e_1)$.

Assume by contradiction that $N^+_{\omega'}(e_1) = N^-_{\omega'}(e_1)$ for every skew-signing ω' of (D,ω). By using an induction process, we can deduce, as in the proof of Lemma 3.4, that $N^+_{\omega'}(e_1,\ldots,e_q) = N^-_{\omega'}(e_1,\ldots,e_q)$ for all skew-signings ω' of (D,ω). However,

$$N^+_{\omega'}(e_1,\ldots,e_q) = \begin{cases} \omega(\overrightarrow{C}_0) + \omega(\overrightarrow{C}^*_0) & \text{if } q \text{ is even and } \omega'(\overrightarrow{C}_0) > 0 \\ 0 & \text{if } q \text{ is even and } \omega'(\overrightarrow{C}_0) < 0 \\ \omega(\overrightarrow{C}_0) & \text{if } q \text{ is odd and } \omega'(\overrightarrow{C}_0) > 0 \\ \omega(\overrightarrow{C}^*_0) & \text{if } q \text{ is odd and } \omega'(\overrightarrow{C}_0) < 0 \end{cases}$$

and

$$N^-_{\omega'}(e_1,\dots,e_q)=\begin{cases}0 & \text{if } q \text{ is even and } \omega'(\overrightarrow{C}_0)>0\\ \omega(\overrightarrow{C}_0)+\omega(\overrightarrow{C}^*_0) & \text{if } q \text{ is even and } \omega'(\overrightarrow{C}_0)<0\\ \omega(\overrightarrow{C}^*_0) & \text{if } q \text{ is odd and } \omega'(\overrightarrow{C}_0)>0\\ \omega(\overrightarrow{C}_0) & \text{if } q \text{ is odd and } \omega'(\overrightarrow{C}_0)<0\end{cases}$$

which contradicts our assumption on $\overrightarrow{C}_0$. This completes the proof of Step 1.

Step 2. (D,ω) is $(\leq q-1)$-cycle-symmetric and contains no even cycles of length $k\in\{3,\dots,q-1\}$.

This follows from the choice of q and Lemma 3.5.

Consider now the skew-signing ω'' of (D,ω) that coincides with ω'_0 outside $\{e_1,e_1^*\}$ and such that $\omega''(e)=-\omega'_0(e)$ for $e\in\{e_1,e_1^*\}$. Let $p_{(D,\omega'_0)}(x):=x^n+b_1x^{n-1}+\cdots+b_{n-1}x+b_n$ and $p_{(D,\omega'')}(x):=x^n+c_1x^{n-1}+\cdots+c_{n-1}x+c_n$ be the characteristic polynomials of (D,ω'_0) and (D,ω'') respectively.

As in the proof of Lemma 3.5, we have

$$\begin{aligned} b_q-c_q &= -2(\eta^+_{\omega'_0}(e_1)+\eta^+_{\omega'_0}(e_1^*))+2(\eta^-_{\omega'_0}(e_1)+\eta^-_{\omega'_0}(e_1^*))\\ &= -2(N^+_{\omega'_0}(e_1)-N^-_{\omega'_0}(e_1))\neq 0\end{aligned}$$

which contradicts Step 1. This ends the proof of lemma. □

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