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On the structure of conservation laws of (3+1)-dimensional wave equation

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Abstract. In this paper, a (3+1)-dimensional wave equation is studied from the point of view of Lie's theory in partial differential equations including conservation laws. The symmetry operators are determined to find the reduced form of the considered equation. The non-local conservation theorems and multipliers approach are performed on the (3+1)-dimensional wave equation. We obtain conservation laws by using five methods, such as direct method, Noether's method, extended Noether's method, Ibragimov's method; and finally we can derive infinitely many conservation laws from a known conservation law viewed as the last method. We also derive some exact solutions using some conservation laws Anco and Bluman (2002).

Keywords: Wave equation; Conservation laws; Lie symmetry; Direct method; Noether's theorem; Boyer's formulation

Mathematics Subject Classification: 76M60; 70H33; 35J05

1. INTRODUCTION

Many partial differential equations (PDEs) of physical importance have not a general theory for finding solutions. While there is no existing general theory for solving such equations the methods of point transformations are a powerful tool. One of the most useful

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point transformations are those which form a continuous group. Lie classical symmetries admitted by a system of PDEs are useful for finding invariant solutions [3,4]. The classical symmetry method for differential equations (DEs) is based on Lie group symmetries. Lie symmetries provide a powerful and systematic tool for analysis of PDEs. For instance, the method of reduction of variables via Lie point symmetries is an extremely useful technique for simplifying or solving PDE's.

The symmetry group of a system of DEs transforms solutions of the system to other solutions of the system. Another important class of symmetries that generalize contact transformations is the class of higher-order symmetries.

A PDEs system can be considered not only as itself but together with its prolongations to all orders. The transformations that preserve the contact system in the infinite jet space and leave the infinite prolongation of the system invariant are called higher-order symmetries; their calculation and possible outcomes upon the wave equation are realized in section two.

A conservation law of a given DEs system is a divergence expression that vanishes on all solutions of the DEs system. In the study of systems of DEs, the concept of a conservation law plays significant roles, not only in obtaining in-depth understanding of physical properties of various systems, but also in the construction of their exact solutions. They describe physical conserved quantities such as mass, energy, momentum and angular momentum, as well as charge and other constants of motion. They are important for investigating integrability and linearization mappings and for establishing existence and uniqueness of solutions. They are also used in the analysis of stability and global behavior of solutions [5,15,8,14]. This work is organized as follows: in Section 3, the conservation laws associated to the wave equation are computed via direct method and other methods. Section 4 is devoted to the construction of exact solutions by utilizing known conservation laws. Finally the conclusions are presented in Section 5.

The general wave equation is an important second-order nonlinear PDE

$$u_{tt} - (f(u)u_x)_x - (g(u)u_y)_y - (h(u)u_z)_z = 0, \tag{1}$$

for the description of waves as they occur in physics such as sound waves, light waves and water waves. It arises in fields like acoustics, electromagnetics, and fluid dynamics. In this paper we study Eq. (1) for $f(u) = g(u) = h(u) = 1$ which takes the form

$$u_{tt} - u_{xx} - u_{yy} - u_{zz} = 0. \tag{2}$$

Then the wide range of solution with Lie symmetry method and conservation are given.

2. Lie symmetries of the equation

It is known that to find exact solutions of PDEs is always one of the central themes in mathematics and physics. Some of the most important methods for finding exact solutions of PDEs are the Lie symmetry analysis [17,19,18,6,12,11].

First of all, let us consider a one-parameter Lie group of infinitesimal transformation:

$$\begin{aligned}
t &= t + \xi_1(t, x, y, z, u) + O(\varepsilon^2),\\
x &= x + \xi_2(t, x, y, z, u) + O(\varepsilon^2),\\
y &= y + \xi_3(t, x, y, z, u) + O(\varepsilon^2),\\
z &= z + \xi_4(t, x, y, z, u) + O(\varepsilon^2),\\
u &= u + \phi(t, x, y, z, u) + O(\varepsilon^2)
\end{aligned} \tag{3}$$

with a small parameter $\varepsilon \ll 1$. The vector field associated with the above group of transformations can be written as

$$X = \xi_1 \frac{\partial}{\partial t} + \xi_2 \frac{\partial}{\partial x} + \xi_3 \frac{\partial}{\partial y} + \xi_4 \frac{\partial}{\partial z} + \phi \frac{\partial}{\partial u}. \tag{4}$$

The operator (4) has the second prolongation

$$\begin{aligned} X^{(2)} = X &+ \phi^t \frac{\partial}{\partial u_t} + \phi^x \frac{\partial}{\partial u_x} + \phi^y \frac{\partial}{\partial u_y} + \phi^z \frac{\partial}{\partial u_z} \\ &+ \phi^{tt} \frac{\partial}{\partial u_{tt}} + \phi^{tx} \frac{\partial}{\partial u_{tx}} + \phi^{ty} \frac{\partial}{\partial u_{ty}} + \phi^{tz} \frac{\partial}{\partial u_{tz}} + \phi^{xx} \frac{\partial}{\partial u_{xx}} \\ &+ \phi^{xy} \frac{\partial}{\partial u_{xy}} + \phi^{xz} \frac{\partial}{\partial u_{xz}} + \phi^{yy} \frac{\partial}{\partial u_{yy}} + \phi^{yz} \frac{\partial}{\partial u_{yz}} + \phi^{zz} \frac{\partial}{\partial u_{zz}}. \end{aligned} \tag{5}$$

Acting (5) on Eq. (2) and using the invariance condition, yields the full symmetry group of the equation spanned by the following sixteen vector fields:

$$\begin{aligned} &X_1 = \frac{\partial}{\partial x}, \qquad X_2 = \frac{\partial}{\partial y}, \qquad X_3 = \frac{\partial}{\partial z}, \qquad X_4 = \frac{\partial}{\partial t}, \\ &X_5 = u\frac{\partial}{\partial u}, \qquad X_6 = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}, \qquad X_7 = t\frac{\partial}{\partial y} + y\frac{\partial}{\partial t}, \\ &X_8 = z\frac{\partial}{\partial t} + t\frac{\partial}{\partial z}, \qquad X_9 = t\frac{\partial}{\partial x} + x\frac{\partial}{\partial t}, \qquad X_{10} = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} \\ &X_{11} = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}, \qquad X_{12} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + t\frac{\partial}{\partial t}, \\ &X_{13} = \frac{1}{2}(y^2 + z^2 - x^2 - t^2)\frac{\partial}{\partial x} - xy\frac{\partial}{\partial y} - xz\frac{\partial}{\partial z} - xt\frac{\partial}{\partial t} - xu\frac{\partial}{\partial u}, \\ &X_{14} = zx\frac{\partial}{\partial x} + zy\frac{\partial}{\partial y} + \frac{1}{2}(t^2 + z^2 - x^2 - y^2)\frac{\partial}{\partial z} + zt\frac{\partial}{\partial t} - zu\frac{\partial}{\partial u}, \\ &X_{15} = yx\frac{\partial}{\partial x} + \frac{1}{2}(y^2 + t^2 - x^2 - z^2)\frac{\partial}{\partial y} + yz\frac{\partial}{\partial z} + yt\frac{\partial}{\partial t} - yu\frac{\partial}{\partial u}, \\ &X_{16} = tx\frac{\partial}{\partial x} + ty\frac{\partial}{\partial y} + tz\frac{\partial}{\partial z} + \frac{1}{2}(x^2 + t^2 - y^2 - z^2)\frac{\partial}{\partial t} - tu\frac{\partial}{\partial u}. \end{aligned} \tag{6}$$

3. Local conservation laws

Conservation laws are one of the most important gateways to understanding physical properties of various systems. They have been used for the development of appropriate numerical methods and construction of exact solutions of PDEs [13,7,20]. The wave equation is the best known example of a non-linear PDE that can be directly transformed to a linear equation. For this reason, and due to its wide range of applications, several studies have been made of generalizations of the wave equation in two or three spatial dimensions. Among these generalizations is the $(3 + 1)$-dimensional equation (2).

3.1. Direct method

This method is an effective algorithmic method which is presented for finding the local conservation laws for PDEs with any number of independent and dependent variables. The method does not require the use or existence of a variational principle and reduces the calculation of conservation laws for solving a system of linear determining. All derived solutions for this system, yields a conservation law for each solution of the determining system.

We review this method and the underlying theory in this section. Within this method, one seeks a set of local multipliers (also called factors or characteristics) depending on independent and dependent variables of a given PDEs system and derivatives of dependent variables up to some fixed order, such that a linear combination of the PDEs of the system which is taken with these multipliers yields a divergence expression. Families of multipliers that yield conservation laws are found from determining equations that follow from Euler differential operators. After finding sets of local conservation law multipliers, one needs to derive expressions for the corresponding conservation law, e.g. fluxes, [1,2].

Consider a system $R\{x;u\}$ of N PDEs of order k with $n-$independent variables $x = (x^1,\dots,x^n)$ and $m-$dependent variables $u(x) = (u^1(x),\dots,u^m(x))$, given by

$$R^\sigma[u] = R^\sigma(x,u,\partial u,\dots,\partial^k u) = 0, \qquad \sigma = 1,\dots,N. \tag{7}$$

Definition 3.1. A local conservation law of PDE system (7) is a divergence expression

$$D_i\phi^i[u] = D_1\phi^1[u]+,\dots,+D_n\phi^n[u] = 0, \tag{8}$$

holding for all solutions of PDE system (7). In (8), $\phi^i[u] = \phi^i(x,u,\partial u,\dots,\partial^r u), i = 1,\dots,n$, are called fluxes of the conservation law and D_i is the total derivatives with respect to x_i.

In general, for a given PDE system (7), nontrivial local conservation laws arise from linear combination of the equations of the PDE system (7) with multipliers (characteristics) that yield nontrivial divergence expressions.

In particular, a set of multipliers $\{\Lambda_\sigma[U]\}_{\sigma=1}^N = \{\Lambda_\sigma(x,U,\partial U,\dots,\partial^l U)\}_{\sigma=1}^N$yields a divergence expression for the PDE system $R\{x;u\}$ if the identity

$$\Lambda_\sigma[U]R^\sigma[U] \equiv D_i\phi^i[U] \tag{9}$$

holds for arbitary functions $U(x)$. Then on the solutions $U(x) = u(x)$ of the PDE system (7), if $\Lambda_\sigma[U]$ is non-singular, one has a local conservation law

$$\Lambda_\sigma[u]R^\sigma[u] = D_i\phi^i[u] = 0. \tag{10}$$

Definition 3.2. The Euler operator with respect to U^j is the operator defined by

$$E_{U^j} = \frac{\partial}{\partial U^j} - D_i\frac{\partial}{\partial U_i^j} + \cdots + (-1)^s D_{i_1\dots i_s}\frac{\partial}{\partial U_{i_1\dots i_s}^j} + \cdots, \qquad j = 1,\dots,m. \tag{11}$$

By direct calculation, one can show that the Euler operator (11) annihilates any divergence expression $D_i\phi^i[U]$. In particular, the following identities for arbitary $U(x)$ hold:

$$E_{U^j}\left(D_i\phi^i(x,U,\partial U,\dots,\partial^r U)\right) \equiv 0, \qquad j = 1,\dots,m. \tag{12}$$

The converse also holds. Specifically, the only scalar expressions annihilated by Euler operator are divergence expressions.

Theorem 3.3. *A set of non-singular local multipliers* $\{\Lambda_\sigma(x, U, \partial U, \ldots, \partial^l U)\}_{\sigma=1}^N$ *yields a local conservation law for the PDE system* $R\{x; u\}$ *if and only if the set of identities*

$$E_{U^j}\left(\Lambda_\sigma(x, U, \partial U, \ldots, \partial^l U) R^\sigma(x, U, \partial U, \ldots, \partial^k U)\right) \equiv 0, \qquad j = 1, \ldots, m. \quad (13)$$

holds for arbitary functions $U(x)$.

Following from Theorem 3.3 a systematic for the construction of local conservation laws, referred to as the direct method, is now outlined.

(I) For a given kth-order PDE system $R\{x; u\}$ (7), seek sets of multipliers of the form $\{\Lambda_\sigma(x, U, \partial U, \ldots, \partial^l U)\}_{\sigma=1}^N$ to some specified order l.
(II) Solve the set of determining equations (13) for arbitary $U(x)$ to find all such sets of multipliers.
(III) Find the corresponding fluxes $\phi^i(x, U, \partial U, \ldots, \partial^r U)$ satisfying the identity

$$\begin{aligned}&\Lambda_\sigma(x, U, \partial U, \ldots, \partial^l U) R^\sigma(x, U, \partial U, \ldots, \partial^k U)\\ &\quad \equiv D_i \phi^i(x, U, \partial U, \ldots, \partial^r U).\end{aligned} \quad (14)$$

(IV) Each set of fluxes and multipliers yields a local conservation law $D_i\phi^i(x, u, \partial u, \ldots, \partial^r u)$, holding for all solutions $u(x)$ of the given PDE system $R\{x; u\}$ (7).

Example 3.4. As an example, regard linear equation (2). We seek all conservation law multipliers of the form

$$\Lambda = \xi(x, y, z, t, U, U_x, U_y, U_z, U_t) \quad (15)$$

of this equation. In the Euler operator

$$\begin{aligned}E_U = {}&\frac{\partial}{\partial U} - D_x\frac{\partial}{\partial U_x} - D_y\frac{\partial}{\partial U_y} - D_z\frac{\partial}{\partial U_z} - D_t\frac{\partial}{\partial U_t}\\ &+ D_{xx}\frac{\partial}{\partial U_{xx}} + D_{yy}\frac{\partial}{\partial U_{yy}} + D_{zz}\frac{\partial}{\partial U_{zz}} + D_{tt}\frac{\partial}{\partial U_{tt}}\end{aligned} \quad (16)$$

the determining equation (13) for multiplier (15) becomes

$$E_U = \left[\xi(x, y, z, t, U, U_x, U_y, U_z, U_t)(u_{tt} - u_{xx} - u_{yy} - u_{zz})\right] \equiv 0. \quad (17)$$

By utilizing softwares such as mathematica and maple we obtain one order multiplier in the following form,

$$\begin{aligned}&\Lambda_1 = u_t, && \Lambda_2 = u_x,\\ &\Lambda_3 = u_y, && \Lambda_4 = u_z,\\ &\Lambda_5 = tu_x + xu_t, && \Lambda_6 = tu_y + yu_t,\\ &\Lambda_7 = -xu_y + yu_x, && \Lambda_8 = tu_z + zu_t,\end{aligned}$$

Table 1
Multiplier and conservation laws applying direct method.

$\Lambda_{(i)}$	$\psi^t[u]$	$\phi^x[u]$	$\phi^y[u]$	$\phi^z[u]$
$\Lambda_{(1)}$	A	$-u_x u_t$	$-u_y u_t$	$-u_z u_t$
$\Lambda_{(2)}$	$u_x u_t$	B	$-u_y u_x$	$-u_z u_x$
$\Lambda_{(3)}$	$u_y u_t$	$-u_x u_y$	C	$-u_z u_y$
$\Lambda_{(4)}$	$u_z u_t$	$-u_x u_z$	$-u_y u_z$	D
$\Lambda_{(5)}$	$tu_x u_t + xA$	$-xu_x u_t - xB$	$-u_y\,(tu_x + xu_t)$	$-u_z\,(tu_x + xu_t)$
$\Lambda_{(6)}$	$tu_y u_t + yA$	$-u_x\left(tu_y + yu_t\right)$	$-yu_y u_t + tC$	$-u_z\left(tu_y + yu_t\right)$
$\Lambda_{(7)}$	$u_t\left(yu_x - xu_y\right)$	$xu_x u_y + yB$	$-yu_y u_x - xC$	$-u_z\left(yu_x - xu_y\right)$
$\Lambda_{(8)}$	$tu_z u_t + zA$	$-u_x\,(tu_z + zu_t)$	$-u_y\,(tu_z + zu_t)$	$-zu_z u_t + tD$

$$
\begin{aligned}
\Lambda_9 &= u_x\left(\frac{y^2+z^2-t^2-x^2}{2}\right) - x(tu_t + yu_y + zu_z + u),\\
\Lambda_{10} &= u_z\left(\frac{t^2+z^2-z^2-x^2}{2}\right) + z(tu_t + yu_y + zu_z + u),\\
\Lambda_{11} &= u_y\left(\frac{y^2+t^2-z^2-x^2}{2}\right) + y(tu_t + yu_y + zu_z + u),\\
\Lambda_{12} &= u_t\left(\frac{y^2+z^2+t^2+x^2}{2}\right) + t(tu_t + yu_y + zu_z + u).
\end{aligned}
\tag{18}
$$

So we stratificate all of one order multiplier accompanied by their conservation laws in Table 1,

where

$$
A = \frac{1}{2}(u_t^2 + u_x^2 + u_y^2 + u_z^2) \quad B = \frac{1}{2}(u_y^2 + u_z^2 - u_x^2 - u_t^2)
$$

$$
C = \frac{1}{2}(u_x^2 + u_z^2 - u_y^2 - u_t^2) \quad D = \frac{1}{2}(u_y^2 + u_x^2 - u_z^2 - u_t^2).
$$

3.2. Noether's method

There are many methods of constructing conservation laws for DEs; A systematic way of constructing the conservation laws of a system of DEs that admits a variational principle is via Noether's theorem [1,16]. Its application allows physicists to gain powerful insights into any general theory in physics just by analyzing the various transformations that would make the form of the laws involved invariant. For instance, the invariance of physical systems with respect to spatial translation, rotation, and time translation respectively give rise to the well known conservation laws of linear momentum, angular momentum and energy. Although Noether's theorem provides an elegant approach to find conservation laws, it possesses a strong limitation: it can only be applied to equations with variational structure. In fact, Noether's method gives an explicit formula for determining a conservation law once a Noether symmetry associated with a Lagrangian is known for an Euler–Lagrange equation. Thus, the knowledge of a Lagrangian is essential in this method.

3.2.1. Self-adjoint PDE systems

Consider a system (7), for an arbitary function $U(x) = (U^1(x), \ldots, U^m(x))$.

The linearizing operator $\mathcal{L}[U]$ associated with the PDEs system (7) is given by

$$\mathcal{L}^{\sigma}_{\rho}[U]V^{\rho} = \left[\frac{\partial R^{\sigma}[U]}{\partial U^{\rho}} + \frac{\partial R^{\sigma}[U]}{\partial U^{\rho}_{i}}D_i + \cdots + \frac{\partial R^{\sigma}[U]}{\partial U^{\rho}_{i_1\ldots i_k}}D_{i_1}\ldots D_{i_k}\right]V^{\rho},$$
$$\sigma = 1,\ldots,N \tag{19}$$

in terms of an arbitary function $V(x) = \big(V^1(x),\ldots,V^m(x)\big)$. The adjoint operator $\mathcal{L}^{\star}[U]$ associated with the PDE system (7) is obtained formally through integration by parts and is given by

$$\mathcal{L}^{\star\sigma}_{\rho}[U]W_{\sigma} = \frac{\partial R^{\sigma}[U]}{\partial U^{\rho}}W_{\sigma} - D_i\left(\frac{\partial R^{\sigma}[U]}{\partial U^{\rho}_{i}}W_{\sigma}\right) + \cdots$$
$$+ (-1)^k D_{i_1}\ldots D_{i_k}\left(\frac{\partial R^{\sigma}[U]}{\partial U^{\rho}_{i_1\ldots i_k}}W_{\sigma}\right), \qquad \varphi = 1,\ldots,m, \tag{20}$$

in terms of an arbitary function $W(x) = (W_1(x),\ldots,W_N(x))$.

Definition 3.5. Let $\mathcal{L}[U]$, with its components $\mathcal{L}^{\sigma}_{\rho}[U]$ given by (19), be the linearizing operator associated with a PDE system (7). The adjoint operator of $\mathcal{L}[U]$ is $\mathcal{L}^{*}[U]$, with its components $\mathcal{L}^{\star\sigma}_{\rho}[U]$ given by (20). $\mathcal{L}[U]$ is a self-adjoint operator if and only if $\mathcal{L}[U] = \mathcal{L}^{*}[U]$, i.e., $\mathcal{L}^{\sigma}_{\rho}[U] = \mathcal{L}^{\star\sigma}_{\rho}[U]$, $\sigma,\rho = 1,\ldots,m$.

Consider a functional $J[U]$ in terms of n independent variables $x = (x^1,\ldots,x^n)$ and m arbitary functions $U = (U^1(x),\ldots,U^m(x))$ and their partial derivatives up to order k, defined on a domain Ω,

$$J[U] = \int_{\Omega}\mathcal{L}[U]\mathrm{d}x = \int_{\Omega}\mathcal{L}(x,U,\partial U,\ldots,\partial^k[U])\mathrm{d}x. \tag{21}$$

The functional $\mathcal{L}[U] = \mathcal{L}(x,U,\partial U,\ldots,\partial^k U)$ is called *Lagrangian* and the functional $J[U]$ is called *action integral*.

We now present Noether's formulation of her theorem. In this formulation, the action integral $J[U]$ is required to be invariant under a one-parameter Lie group of point transformations

$$\tilde{x}^i = x^i + \varepsilon\xi^i(x,U) + \mathrm{O}(\varepsilon^2), \qquad i = 1,\ldots,n,$$
$$\tilde{U}^{\alpha} = U^{\alpha} + \varepsilon\eta^{\alpha}(x,U) + \mathrm{O}(\varepsilon^2), \qquad \alpha = 1,\ldots,m, \tag{22}$$

with the corresponding kth extended infinitesimal generator $\tilde{X}$. It is shown that the one-parameter Lie group of point transformation (22) is equivalent to the one-parameter family of local transformation

$$\tilde{x}^i = x^i, \qquad i = 1,\ldots,n,$$
$$\tilde{u}^{\alpha} = u^{\alpha} + \varepsilon\tilde{\eta}^{\alpha}(x,u^{(k)}) + \mathrm{O}(\varepsilon^2), \qquad \alpha = 1,\ldots,q, \tag{23}$$

over which $v^{\alpha}(x) = \tilde{\eta}^{\alpha}[U] = \eta^{\alpha}(x,U) - U^{\alpha}_{i}\xi^i(x,U)$ and with the corresponding kth extended infinitesimal generator $\tilde{X^{(k)}}$ given through the appropriate truncation of following expression [4],

$$\tilde{X}^{(k)} = \sum_{\alpha=1}^{q}[\eta^{\alpha}(x,U) - \sum_{i=1}^{p}u^{\alpha}_{i}\xi^i(x,U)]\frac{\partial}{\partial u^{\alpha}}.$$

Theorem 3.6. *Suppose a given PDE system, as written, arises from a variational principle, i.e., the given PDE system is a set of Euler–Lagrange equations*

$$E_{u^\sigma}(\mathcal{L}[U]) = \left(\frac{\partial \mathcal{L}[U]}{\partial U^\sigma} + \cdots + (-1)^{(k)} D_{j_1} \ldots D_{j_{(k)}} \frac{\partial \mathcal{L}[U]}{\partial U^\sigma_{j_1 \cdots j_k}}\right) = 0,$$
$$\sigma = 1, \ldots, m, \tag{24}$$

whose solutions $u(x)$ are extrema $U(x) = u(x)$ of an action integral (21) with Lagrangian $\mathcal{L}[U]$. Suppose the one-parameter Lie group of point transformation (22) is a point symmetry of $J[U]$. Let $W^i[U, v]$ be defined by

$$\begin{aligned} W^i[U, v] &= v^\sigma \left(\frac{\partial \mathcal{L}[U]}{\partial U^\sigma_i} + \cdots + (-1)^{(k-1)} D_{j_1} \ldots D_{j_{(k-1)}} \frac{\partial \mathcal{L}[U]}{\partial U^\sigma_{ij_1 \cdots j_{k-1}}}\right) \\ &\quad + v^\sigma_{ij_1} \left(\frac{\partial \mathcal{L}[U]}{\partial U^\sigma_{ij_1}} + \cdots + (-1)^{(k-2)} D_{j_2} \ldots D_{j_{(k-1)}} \frac{\partial \mathcal{L}[U]}{\partial U^\sigma_{ij_1 \cdots j_{k-1}}}\right) \\ &\quad + \cdots + v^\sigma_{j_1 \cdots j_{(k-1)}} \left(\frac{\partial \mathcal{L}[U]}{\partial U^\sigma_{ij_1 \cdots j_{k-1}}}\right) \end{aligned} \tag{25}$$

for arbitary functions $U(x)$, $v(x)$. Then

- *The identity*

$$\tilde{\eta}^\alpha[U] E_{U^\alpha}(\mathcal{L}[U]) \equiv -D_i \left(\xi^i(x, U)\mathcal{L}[U] + W^i[U, \tilde{\eta}^\alpha[U]]\right) \tag{26}$$

  holds for arbitary functions $U(x)$.
- *The local conservation laws*

$$D_i \left(\xi^i(x, U)\mathcal{L}[U] + W^i[U, \tilde{\eta}[U]]\right) = 0, \tag{27}$$

  hold for any solution $u = \Theta(x)$ of the Euler–Lagrange system (24).

Example 3.7. Consider Eq. (2) with Lagrangian $\mathcal{L}[U] = \frac{1}{2}(u_x^2 + u_y^2 + u_z^2 - u_t^2)$. The evolution form of the infinitesimal generator X_1 is

$$\tilde{X}_1 = -u_x \frac{\partial}{\partial u}$$

and once extended the infinitesimal generator (the one order prolongation of $\tilde{X}_1$) is given by

$$\tilde{X}_1^{(1)} = -u_x \frac{\partial}{\partial u} - u_{xx} \frac{\partial}{\partial u_x} - u_{xy} \frac{\partial}{\partial u_y} - u_{xz} \frac{\partial}{\partial u_z} - u_{xt} \frac{\partial}{\partial u_t}. \tag{28}$$

The action of the extension (28) on the Lagrangian $\mathcal{L}[U]$ yields the divergence expression

$$\tilde{X}_1^{(1)} \mathcal{L}[U] = -D_x \left(\frac{u_x^2 + u_y^2 + u_z^2 - u_t^2}{2}\right).$$

Hence Noether's theorem yields density ψ^t and fluxes ϕ^x, ϕ^y and ϕ^z such as

$$\psi^t = u_x u_t, \quad \phi^x = \frac{u_z^2 + u_y^2 - u_x^2 - u_t^2}{2}, \quad \phi^y = -u_x u_y, \quad \phi^z = -u_x u_z.$$

Table 2
Conservation laws applying Noether's method.

X_i	$\psi^t[u]$	$\phi^x[u]$	$\phi^y[u]$	$\phi^z[u]$
X_1	$u_x u_t$	B	$-u_x u_y$	$-u_x u_z t$
X_2	$-u_y u_t$	$-u_x u_y$	B	$-u_y u_z$
X_3	$-u_z u_t$	$-u_x u_z$	$-u_y u_z$	B
X_4	B	$-u_x u_t$	$-u_y u_t$	$-u_z u_t$
X_6	$-u_t(yu_z - zu_y)$	$u_x(yu_z - zu_y)$	$u_y(yu_z - zu_y) + zB$	$u_z(yu_z - zu_y) - yB$
X_7	$u_t(tu_y + yu_t) + yB$	$-u_x(tu_y + yu_t)$	$-u_y(tu_y + yu_t) + tB$	$-u_z(tu_y + yu_t)$
X_8	$u_t(tu_z + zu_t) + zB$	$-u_x\,(tu_z + zu_t)$	$-u_y\,(tu_z + zu_t)$	$-u_z(tu_z + zu_t) + t\left(\frac{u_x^2+u_y^2+u_z^2-u_t^2}{2}\right)$
X_9	$u_t(tu_x + xu_t) + xB$	$-u_x\,(tu_x + xu_t) + tB$	$-u_y\,(tu_x + xu_t)$	$-u_z(tu_x + xu_t)$
X_{10}	$-u_t(xu_z - zu_x)$	$u_x(xu_z - zu_x) + zB$	$u_y(xu_z - zu_x)$	$u_z(xu_z - zu_x) - xB$
X_{11}	$-u_t(xu_y - yu_x)$	$u_x(xu_y - yu_x) + yB$	$u_y(xu_y - yu_x) - xB$	$u_z(xu_y - yu_x)$
X_{12}	$u_t A + tB$	$-u_x A + xB$	$-u_y A + yB$	$-u_z(xu_x + yu_y + zu_z + tu_t) + zB$
X_{13}	$-u_t C - xtB$	$u_x C - \left(\frac{x^2+t^2-y^2-z^2}{2}\right) B$	$u_y C - xyB$	$u_z C - xzB$
X_{14}	$-u_t D + ztB$	$u_x D + zxB$	$u_y D + zyB$	$u_z D + \left(\frac{z^2+t^2-x^2-y^2}{2}\right) B$
X_{15}	$-u_t E + ytB$	$u_x E + yxB$	$u_y E + \left(\frac{y^2+t^2-x^2-z^2}{2}\right) B$	$u_z E + yzB$
X_{16}	$-u_t F + \left(\frac{x^2+t^2-y^2-z^2}{2}\right) B$	$u_x F + xtB$	$u_y F + ytB$	$u_z F + ztB$

We collect the other generators in Table 2 where

$$
\begin{aligned}
A &= (xu_x + yu_y + zu_z + tu_t),\\
B &= \frac{u_x^2 + u_y^2 + u_z^2 - u_t^2}{2},\\
C &= xu + \left(\frac{x^2 + t^2 - y^2 - z^2}{2}\right) u_x + xyu_y + xzu_z + xtu_t,\\
D &= -zu - zxu_x - zyu_y - \left(\frac{z^2 + t^2 - y^2 - x^2}{2}\right) u_z - ztu_t,\\
E &= -yu - yxu_x - \left(\frac{y^2 + t^2 - x^2 - z^2}{2}\right) u_y - yzu_z - ytu_t,\\
F &= -tu - txu_x - tyu_y - tzu_z - \left(\frac{x^2 + t^2 - y^2 - z^2}{2}\right) u_t.
\end{aligned}
$$

3.3. Boyer's formulation of Noether's theorem (Extended Noether's theorem method)

Boyer [6] extended Noether's theorem to enable one to conveniently find conservation laws arising from invariance under higher-order transformations by generalizing Noether's definition of invariance of an action integral $J[U]$. In particular, under the following definition, an action integral $J[U]$ is invariant under a one-parameter higher-order transformation if its integrand $\mathcal{L}[U]$ is in divergence form under such a transformation [1,5]. This terminology is cleared in the sequel.

Definition 3.8. Let

$$\tilde{X} = \tilde{\eta^\alpha}(x, U, \partial U, \ldots, \partial^s U)\frac{\partial}{\partial U^\alpha}, \tag{29}$$

be the infinitesimal generator of a one-parameter higher-order local transformation

$$(x^*)^i = x^i, \qquad i = 1, \ldots, n,$$
$$(u^*)^\alpha = \exp(\varepsilon \tilde{X}^\infty)u^\alpha = u^\alpha + \sum_{j=1}^{\infty} \frac{\varepsilon^j}{j!}(\tilde{X}^\infty)^{j-1}\tilde{\eta}^\alpha, \qquad \alpha = 1, \ldots, m,$$

with its extension $\tilde{X}^{(\infty)}$ given by

$$\tilde{X}^{(\infty)} = \tilde{\eta}^\alpha \frac{\partial}{\partial u^\alpha} + \tilde{\eta}_i^{(1)\alpha} \frac{\partial}{\partial u_i^\alpha} + \cdots + \tilde{\eta}_{i_1 \ldots i_p}^{(p)\alpha} \frac{\partial}{\partial u_{i_1 \ldots i_p}^\alpha} + \cdots . \tag{30}$$

Let $\tilde{\eta}^\alpha = \tilde{\eta}^\alpha(x, U.\partial U, \ldots, \partial^s U)$. The transformation is a local symmetry of (21) if and only if

$$\tilde{X}^{(\infty)}\mathcal{L}[U] \equiv D_i A^i[U],$$

holds for some set of functions $A^i[U] = A^i(x, U, \partial U, \ldots, \partial^r U), \qquad i =, \ldots, n.$

Theorem 3.9. *Suppose a given PDE system $R\{x; u\}$, as written, arises from a variational principle, i.e., the given PDE system is a set of Euler–Lagrange equations (24) whose solutions $u(x)$ are extrema $U(x) = u(x)$ of an action integral (21) with Lagrangian $\mathcal{L}[U]$. Suppose that a local transformation with infinitesimal generator (29) yields a variational symmetry of (21). Let $W^i[U, v]$ be defined by (25) for arbitrary functions $U(x)$ and $v(x)$. Then*

- *The identity*

$$\tilde{\eta}^\alpha[U]E_{U^\alpha}(\mathcal{L}[U]) \equiv D_i(A^i[U] - W^i[U, \tilde{\eta}[U]]), \tag{31}$$

holds for arbitary functions $U(x)$, i.e., $\{\tilde{\eta^\alpha}[U]\}_{\alpha=1}^m$ is a set of local multiplier of the Euler–Lagrange equations (24).
- *The local conservation laws*

$$D_i(W^i[U, \tilde{\eta}[U]] - A^i[U]) = 0, \tag{32}$$

holds for any solution $u = \Theta(x)$ of the Euler–Lagrange system (24).

Example 3.10. Conservation laws with Boyer's formula:

(1) Conservation laws arising from X_1
For this generator we have

$$\tilde{X}_1^\infty \mathcal{L}[U] = D_t(0) - D_x\left(\frac{u_x^2 + u_y^2 + u_z^2 - u_t^2}{2}\right) + D_y(0) + D_z(0),$$

thus, the corresponding conservation laws are

$$D_t(u_x u_t) + D_x\left(\frac{u_y^2 + u_z^2 - u_x^2 - u_t^2}{2}\right) + D_y(-u_x u_y) + D_z(-u_x u_z) = 0.$$

(2) Conservation laws arising from X_2
For this symmetry we have

$$\tilde{X}_2^{\infty}\mathcal{L}[U] = D_t(0) + D_x(0) - D_y\left(\frac{u_x^2 + u_y^2 + u_z^2 - u_t^2}{2}\right) + D_z(0),$$

thus, the related conservation laws are

$$D_t(-u_y u_t) + D_x(-u_x u_y) + D_y\left(\frac{u_x^2 + u_z^2 - u_y^2 - u_t^2}{2}\right) + D_z(-u_y u_z) = 0.$$

(3) Conservation laws arising from X_3
For this operator we have

$$\tilde{X}_3^{\infty}\mathcal{L}[U] = D_t(0) + D_x(0) + D_y(0) - D_z\left(\frac{u_x^2 + u_y^2 + u_z^2 - u_t^2}{2}\right),$$

thus, it leads to the conservation laws

$$D_t(-u_t u_z) + D_x(-u_x u_z) + D_y(-u_z u_y) + D_z\left(\frac{u_x^2 + u_y^2 - u_t^2 - u_z^2}{2}\right) = 0.$$

(4) Conservation laws arising from X_4
For this generator we have

$$\tilde{X}_4^{\infty}\mathcal{L}[U] = -D_t\left(\frac{u_x^2 + u_y^2 + u_z^2 - u_t^2}{2}\right) + D_x(0) + D_y(0) + D_z(0),$$

thus, the corresponding conservation laws are

$$D_t\left(\frac{u_x^2 + u_y^2 + u_z^2 + u_t^2}{2}\right) + D_x(-u_x u_t) + D_y(-u_y u_t) + D_z(-u_z u_t) = 0.$$

(5) Conservation laws arising from X_6
From this symmetry it takes to

$$\begin{aligned}\tilde{X}_6^{\infty}\mathcal{L}[U] = \; & D_t(0) + D_x(0) - D_y\left(z\frac{u_x^2 + u_y^2 + u_z^2 - u_t^2}{2}\right)\\ & + D_z\left(y\frac{u_x^2 + u_y^2 + u_z^2 - u_t^2}{2}\right),\end{aligned}$$

so, we derive the conservation laws

$$\begin{aligned}& D_t\left(-u_t(yu_z - zu_y)\right) + D_x\left(u_x(yu_z - zu_y)\right)\\ & \quad + D_y\left(u_y(yu_z - zu_y) + z\frac{u_x^2 + u_y^2 + u_z^2 - u_t^2}{2}\right)\\ & \quad + D_z\left(u_z(yu_z - zu_y) - y\frac{u_x^2 + u_y^2 - u_t^2 - u_z^2}{2}\right) = 0.\end{aligned}$$

(6) Conservation laws arising from X_7

For this symmetry we derive

$$\tilde{X}_7^{\infty}\mathcal{L}[U] = -D_t\left(y\frac{u_x^2+u_y^2+u_z^2-u_t^2}{2}\right) + D_x(0) - D_y\left(t\frac{u_x^2+u_y^2+u_z^2-u_t^2}{2}\right) + D_z(0),$$

consequently, the corresponding conservation laws are

$$D_t\left(u_t(tu_y+yu_t) + y\frac{u_x^2+u_y^2+u_z^2-u_t^2}{2}\right) + D_x\left(-u_x(tu_y+yu_t)\right) + D_y\left(-u_y(tu_y+yu_t) + t\frac{u_x^2+u_y^2+u_z^2-u_t^2}{2}\right) + D_z\left(-u_z(tu_y+yu_t)\right) = 0.$$

(7) Conservation laws arising from X_8

For this generator we have

$$\tilde{X}_8^{\infty}\mathcal{L}[U] = -D_t\left(z\frac{u_x^2+u_y^2+u_z^2-u_t^2}{2}\right) + D_x(0) + D_y(0) - D_z\left(t\frac{u_x^2+u_y^2+u_z^2-u_t^2}{2}\right),$$

so, we derive the following conservation laws

$$D_t\left(u_t(tu_z+zu_t) + z\frac{u_x^2+u_y^2+u_z^2-u_t^2}{2}\right) + D_x\left(-u_x(tu_z+zu_t)\right) + D_y\left(-u_y(tu_z+zu_t)\right) + D_z\left(-u_z(tu_z+zu_t) + t\frac{u_x^2+u_y^2+u_z^2-u_t^2}{2}\right) = 0.$$

(8) Conservation laws arising from X_9

For this generator we have

$$\tilde{X}_9^{\infty}\mathcal{L}[U] = -D_t\left(z\frac{u_x^2+u_y^2+u_z^2-u_t^2}{2}\right) + D_x(0) + D_y(0) - D_z\left(t\frac{u_x^2+u_y^2+u_z^2-u_t^2}{2}\right),$$

thus, the corresponding conservation laws are

$$D_t\left(u_t(tu_z+zu_t)+z\frac{u_x^2+u_y^2+u_z^2-u_t^2}{2}\right)+D_x\left(-u_x(tu_z+zu_t)\right)$$
$$+D_y\left(-u_y(tu_z+zu_t)\right)+D_z\left(-u_z(tu_z+zu_t)+t\frac{u_x^2+u_y^2+u_z^2-u_t^2}{2}\right)$$
$$=0.$$

3.4. Nonlinear self-adjointness in constructing conservation laws

Many authors developed some methods which do not rely on the knowledge of Lagrangian functions to obtain conservation laws, such as Ibragimov's method, method of multipliers, variational approach on space of solutions of a DE [10,9] and so on. Recently, Ibragimov has proved a result in [9,13] which allows one to construct conservation laws for equations without variational structure. Essentially, Ibragimov's theorem is an extension of Noether's theorem by introducing formal Lagrangian to get rid of the variational limitation. In this paper, we focus on the nonlinear self-adjointness, Lagrangian and associated conservation laws for Eq. (2). On the one hand, we study the nonlinear self-adjointness of Eq. (2) and obtain a different substitution $v = u$. Consequently, we can construct some new conservation laws of this equation by Ibragimov's method.

The concept of nonlinear self-adjointness has significantly expanded the notion of adjointness with respect to construction of conservation laws. It incorporates all the previous concepts of adjointness and thus enables more conserved vectors for DEs to be constructed. Before we proceed to establish the general nonlinearly self-adjoint conditions for the wave equation, we will like to recall the following definitions that are relevant to our work.

Definition 3.11. The adjoint equations to Eq. (2) are given by

$$F_\alpha^*\left(x,u,v,u_{(1)},v_{(1)},\ldots,u_{(s)},v_{(s)}\right)=0,\qquad \alpha=1,\ldots,m. \tag{33}$$

Here $v = (v^1,\ldots,v^m)$ are new dependent variables, $v_{(1)},\ldots,v_{(s)}$ are their derivatives,e.g., $v_i^\alpha = D_i(v^\alpha)$. We use $\frac{\delta}{\delta u^\alpha}$ for the Euler–Lagrange operator

$$\frac{\delta}{\delta u^\alpha}=\frac{\partial}{\partial u^\alpha}+\sum_{s=1}^{\infty}(-1)^s D_{i_1}\ldots D_{i_s}\frac{\partial}{\partial u^\alpha_{i_1\ldots i_s}}\qquad \alpha=1,\ldots,m. \tag{34}$$

Definition 3.12. The system (7) with m dependent variables $u=(u^1,\ldots,u^m)$ is said to be nonlinear self-adjoint if the adjoint equations

$$F_\alpha^*\left(x,u,v,u_{(1)},v_{(1)},\ldots,u_{(s)},v_{(s)}\right)=\frac{\delta(v^\beta F_\beta)}{\delta u^\alpha},\quad \alpha=1,\ldots,m \tag{35}$$

are satisfied for all solutions u of the original system (7) upon a substitution

$$v^\alpha=\varphi^\alpha(x,u),\quad \alpha=1,\ldots,m\qquad \varphi(x,u)\neq 0. \tag{36}$$

In other words, the following equations hold:

$$F_\alpha^*(x,u,\varphi(x,u),\ldots,u_{(s)},\varphi_{(s)})=\lambda_\alpha^\beta F_\beta(x,u,\ldots,u_s)\quad \alpha=1,\ldots,m, \tag{37}$$

where λ_α^β are undetermined coefficients, and $\varphi_{(\sigma)}(x,u)$ are derivatives of (36)

$$\varphi_\sigma = D_{i_1,\dots,i_\sigma}\left(\varphi^\alpha(x,u)\right) \quad \alpha = 1,\dots,s.$$

Theorem 3.13. *Let the system (7) be nonlinear self-adjoint. Specifically, let the adjoint system (35) to (7) be satisfied for all solutions of Eqs. (7) upon a substitution (36). Then any Lie point, contact or Lie–Backlund symmetry (30) as well as a nonlocal symmetry of Eq. (7) leads to the conservation laws $D_i(C^i) = 0$ constructed by the following formula:*

$$\begin{aligned} C^i = {} & W^\alpha \left[\frac{\partial \mathcal{L}}{\partial u_i^\alpha} - D_j\left(\frac{\partial \mathcal{L}}{\partial u_{ij}^\alpha}\right) + D_j D_k \left(\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha}\right) - \cdots \right] \\ & + D_j(W^\alpha)\left[\frac{\partial \mathcal{L}}{\partial u_{ij}^\alpha} - D_k\left(\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha}\right) + \cdots\right] + D_j D_k (W^\alpha)\left[\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha} - \cdots\right], \end{aligned} \tag{38}$$

where

$$W = \eta^\alpha - \xi^j u_j^\alpha$$

and $\mathcal{L}$ is the formal Lagrangian for the system (7) which is,

$$\mathcal{L} = v^\beta F_\beta. \tag{39}$$

Example 3.14. Let us apply Theorem 3.13 to Eq. (2). The formal Lagrangian (39) for the wave equation has the form

$$\mathcal{L} = v(u_{tt} - u_{xx} - u_{yy} - u_{zz}). \tag{40}$$

Thus, we have

$$\begin{aligned} F^* &= D_{xx}(-v) + D_{yy}(-v) + D_{zz}(-v) + D_{tt}(v)\Big|_{v=u} \\ &= u_{tt} - u_{xx} - u_{yy} - u_{zz} = 0. \end{aligned} \tag{41}$$

Hence, Eq. (2) satisfies the condition (37) for $\lambda = 1$. Thus, it is obvious that the wave equation in Definition 3.11 is satisfied. We conclude that Eq. (2) is nonlinear self-adjoint. Therefore we can use Theorem 3.13 for constructing conservation laws. By utilizing sixteen-dimensional Lie algebra with the basis (6) the conservation equation for Eq. (2) will be written in the form

$$D_t(\psi^t) + D_x(\phi^x) + D_y(\phi^y) + D_z(\phi^z) = 0. \tag{42}$$

For instance if we consider $X_1 = \frac{\partial}{\partial x}$, we have $W = -u_x$. Thus,

$$\begin{aligned} \psi^t &= u_x u_t - u u_{xt}, \\ \phi^x &= u u_{xx} - u_x^2, \\ \phi^y &= u u_{xy} - u_x u_y, \\ \phi^z &= u u_{xz} - u_x u_z. \end{aligned}$$

Other conservation laws are coming in Tables 3 and 4.

Table 3
Conservation laws applying Ibragimov method.

X_1	W	$-u_x$
	ψ^t	$u_x u_t - u u_{xt}$
	ϕ^x	$u u_{xx} - u_x^2$
	ϕ^y	$u u_{xy} - u_x u_y$
	ϕ^z	$u u_{xz} - u_x u_z$
X_2	W	$-u_y$
	ψ^t	$u_y u_t - u u_{yt}$
	ϕ^x	$u u_{xy} - u_x u_y$
	ϕ^y	$u u_{yy} - u_y^2$
	ϕ^z	$u u_{yz} - u_y u_z$
X_3	W	$-u_z$
	ψ^t	$u_z u_t - u u_{zt}$
	ϕ^x	$u u_{xz} - u_x u_z$
	ϕ^y	$u u_{yz} - u_y u_z$
	ϕ^z	$u u_{zz} - u_z^2$
X_4	W	$-u_t$
	ψ^t	$-(u_t^2 + u u_{tt})$
	ϕ^x	$u u_{xt} - u_x u_t$
	ϕ^y	$u u_{yt} - u_y u_t$
	ϕ^z	$u u_{zt} - u_z u_t$
X_5	W	u
	ψ^t	0
	ϕ^x	0
	ϕ^y	0
	ϕ^z	0
X_6	W	$y u_z - z u_y$
	ψ^t	$-u_t W + u(y u_{zt} - z u_{yt})$
	ϕ^x	$-u_x W - u(y u_{xz} - z u_{xy})$
	ϕ^y	$-u_y W - u(y u_{yz} - z u_{yy} + u_z)$
	ϕ^z	$-u_z W - u(y u_{zz} - z u_{yz} - u_y)$
X_7	W	$-(y u_t + t u_y)$
	ψ^t	$-u_t W + u(t u_{yt} + y u_{tt} + u_y)$
	ϕ^x	$u_x W + u(t u_{xy} + y u_{xt})$
	ϕ^y	$u_y W + u(t u_{ty} + y u_{tt} + u_t)$
	ϕ^z	$u_z W + u(t u_{zy} + y u_{zt})$
X_8	W	$-(t u_z + z u_t)$
	ψ^t	$-u_t W - u(t u_{zt} + z u_{tt} + u_z)$
	ϕ^x	$-u_x W - u(t u_{xz} + z u_{xt})$
	ϕ^y	$-u_y W - u(t u_{zy} + z u_{yt})$
	ϕ^z	$-u_z W - u(t u_{zz} + z u_{zt} + u_t)$

3.5. Use of symmetries to find new conservation laws from known conservation laws

We derive formula related to obtaining new conservation laws from known conservation laws under the action of an invertible (point or contact) transformation. This formula shows how to use an invertible transformation, including a discrete one, that maps any given PDE system $R\{x; u\}$ to another PDE system $S\{z; w\}$ to obtain directly a conservation law of

Table 4
Conservation laws applying Ibragimov method.

	W	$-(tu_x + xu_t)$
	ψ^t	$-u_t W - u(u_x + tu_{xt} + xu_{tt})$
X_9	ϕ^x	$u_x W + u(u_t + tu_{xx} + xu_{xt})$
	ϕ^y	$u_y W + u(tu_{xy} + xu_{yt})$
	ϕ^z	$u_z W + u(tu_{xz} + xu_{zt})$
	W	$(xu_z - zu_x)$
	ψ^t	$-u_t W + u(xu_{zt} - zu_{xt})$
X_{10}	ϕ^x	$u_x W - u(u_z + xu_{xz} - zu_{xx})$
	ϕ^y	$u_y W - u(xu_{yz} - zu_{xy})$
	ϕ^z	$u_z W - u(xu_{zz} - zu_{xz} - u_x)$
	W	$xu_y - yu_x$
	ψ^t	$-u_t W + u(xu_{yt} - yu_{xt})$
X_{11}	ϕ^x	$u_x W - u(u_y + xu_{xy} - yu_{xx})$
	ϕ^y	$u_y W - u(xu_{yy} - yu_{xy} - u_x)$
	ϕ^z	$u_z W - u(xu_{yz} - yu_{xz})$
	W	$-(xu_x + yu_y + zu_z + tu_t)$
	ψ^t	$-u_t W - u(xu_{xt} + yu_{yt} + zu_{zt} + tu_{tt})$
X_{12}	ϕ^x	$u_x W + u(xu_{xx} + yu_{xy} + zu_{xz} + tu_{xt} + u_x)$
	ϕ^y	$u_y W + u(xu_{xy} + yu_{yy} + zu_{yz} + tu_{yt} + u_y)$
	ϕ^z	$u_z W + u(xu_{xz} + yu_{yz} + zu_{zz} + tu_{zt} + u_z)$
	W	$xu - (\frac{y^2+z^2-x^2-t^2}{2} u_x + xyu_y + xzu_z + xtu_t)$
	ψ^t	$-u_t W + u\left(2xu_t + tu_x + (\frac{y^2+z^2-x^2-t^2}{2})u_{xt} + xyu_{yt} + xzu_{zt} + xtu_{tt}\right)$
X_{13}	ϕ^x	$u_x W - u\left(u + xu_x + yu_y + zu_z + tu_t - (\frac{y^2+z^2-x^2-t^2}{2})u_{xx} + xyu_{xy} + xzu_{xz} + xtu_{xt}\right)$
	ϕ^y	$u_y W - u\left(2xu_y - yu_x - (\frac{y^2+z^2-x^2-t^2}{2})u_{xy} + xyu_{yy} + xzu_{yz} + xtu_{yt}\right)$
	ϕ^z	$u_z W - u\left(2xu_z - zu_x - (\frac{y^2+z^2-x^2-t^2}{2})u_{xz} + xyu_{yz} + xzu_{zz} + xtu_{zt}\right)$
	W	$-(zu + zxu_x + zyu_y + ztu_t + (\frac{x^2+y^2-z^2-t^2}{2})u_z)$
	ψ^t	$-u_t W - u\left(2zu_t - tu_z + xzu_{xt} + yzu_{yt} + ztu_{tt} + (\frac{x^2+y^2-z^2-t^2}{2})u_{zt}\right)$
X_{14}	ϕ^x	$u_x W + u\left(2zu_x + xu_z + xzu_{xx} + yzu_{xy} + ztu_{xt} + (\frac{x^2+y^2-z^2-t^2}{2})u_{xz}\right)$
	ϕ^y	$u_y W + u\left(2zu_y + yu_z + xzu_{xy} + yzu_{yy} + ztu_{yt} + (\frac{x^2+y^2-z^2-t^2}{2})u_{yz}\right)$
	ϕ^z	$u_z W +$ $u\left(u + xu_x + yu_y + tu_t - zu_z + zu_{zz} + xzu_{xz} + yzu_{yt} + xtu_{zt} + (\frac{x^2+y^2-z^2-t^2}{2})u_{zz}\right)$
	W	$-\left(yu + xyu_x + yzu_z + ytu_t + (\frac{y^2+t^2-x^2-z^2}{2})u_y\right)$
	ψ^t	$-u_t W - u\left(2yu_t + tu_y + xyu_{xt} + yzu_{zt} + ytu_{tt} + (\frac{y^2+t^2-x^2-z^2}{2})u_{yt}\right)$
X_{15}	ϕ^x	$u_x W + u\left(2yu_x - xu_y + xyu_{xx} + yzu_{xz} + ytu_{xt} + (\frac{y^2+t^2-x^2-z^2}{2})u_{xy}\right)$
	ϕ^y	$u_y W + u\left(u + xu_x + 2yu_y + tu_t + zu_z + xyu_{xy} + yzu_{zy} + ytu_{yt} + (\frac{y^2+t^2-x^2-z^2}{2})u_{yy}\right)$
	ϕ^z	$u_z W + u\left(2yu_z - zu_y + xyu_{xz} + yzu_{zz} + ytu_{zt} + (\frac{y^2+t^2-x^2-z^2}{2})u_{yz}\right)$
	W	$-\left(tu + xtu_x + ytu_y + ztu_z - (\frac{x^2+t^2-y^2-z^2}{2})u_t\right)$
	ψ^t	$-u_t W - u\left(u + 2tu_t + xu_x + yu_y + zu_z + xtu_{xt} + ytu_{yt} + ztu_{zt} + (\frac{x^2+t^2-y^2-z^2}{2})u_{tt}\right)$

(*continued on next page*)

Table 4 (*continued*)

X_{16}	ϕ^x	$u_x W + u\Big(2tu_x + xu_t + xtu_{xx} + ytu_{xy} + ztu_{xz} + (\frac{x^2+t^2-y^2-z^2}{2})u_{xt}\Big)$
	ϕ^y	$u_y W + u\Big(2tu_y - yu_t + xtu_{xy} + ytu_{yy} + ztu_{zy} + (\frac{x^2+t^2-y^2-z^2}{2})u_{yt}\Big)$
	ϕ^z	$u_x W + u\Big(2tu_z - zu_t + xtu_{xz} + ytu_{yz} + ztu_{zz} + (\frac{x^2+t^2-y^2-z^2}{2})u_{zt}\Big)$

$S\{z; w\}$ from any known conservation law of $R\{x; u\}$. The situation is particularly interesting when the invertible transformation is a symmetry of the given PDE system $R\{x; u\}$ since here one could obtain new conservation laws from a known conservation law of $R\{x; u\}$.

Consider a system of N partial differential equations $R\{x; u\}$ given by (7). Imagine an invertible point transformation

$$\begin{aligned} x^i &= x^i(z, W), \qquad i, \ldots, n, \\ U^\mu &= U^\mu(z, W), \qquad \mu, \ldots, m, \end{aligned} \tag{43}$$

where $U(x) = \big(U^1(x), \ldots, U^m(x)\big)$, $z = \big(z^1, \ldots, z^m\big)$, $W(z) = \big(W^1(z), \ldots, W^m(z)\big)$. Under a point transformation (43) and its natural extensions (prolongations), each function $R^\sigma[U]$ is mapped to some function $S^\sigma[W] = S^\sigma(z, W, \partial W, \ldots, \partial^k W)$. If $U(x) = u(x)$ solves PDE system $R\{x; u\}$(7), then corresponding $W(z) = w(z)$ solves PDE system $S\{z; w\}$ given by

$$S^\sigma[w] = S^\sigma(z, w, \partial w, \ldots, \partial^k w), \qquad \sigma, \ldots, N. \tag{44}$$

Theorem 3.15. *Suppose $D_i\phi^i[u] = 0$ is a conservation law of PDE system (7). Under the point transformation (7), there exist functions $\{\psi^i[W]\}_{i=1}^n$ such that the formula*

$$J[W]D_i\phi^i[U] = \tilde{D}_i\psi^i[W] \tag{45}$$

holds, where $\psi^i[W]$ is given explicitly in terms of the determinant obtained by replacing the ith column of the Jacobian determinant

$$J[W] = \frac{D(x^1, \ldots, x^m)}{D(z^1, \ldots, z^n)} \tag{46}$$

by $[\phi^1[U], \ldots, \phi^n[U]]$ and where D_i, $\tilde{D}_i$ are total derivative operators, respectively, given by

$$\begin{aligned} D_i &= \frac{\partial}{\partial x^i} + U_i^\mu \frac{\partial}{\partial U^\mu} + U_{ii_1}^\mu \frac{\partial}{\partial U_{i_1}^\mu} + \cdots, \\ \tilde{D}_i &= \frac{\partial}{\partial z^i} + W_i^\mu \frac{\partial}{\partial W^\mu} + W_{ii_1}^\mu \frac{\partial}{\partial W_{i_1}^\mu} + \cdots, \qquad i = 1, \ldots, n \end{aligned} \tag{47}$$

with $U_i^\mu = \frac{\partial U^\mu}{\partial x^i}$, $W_i^\mu = \frac{\partial W^\mu}{\partial z^i}$.

3.5.1. Symmetry action on a conservation law to yield new conservation laws

We show that the action of a symmetry on a conservation law of $R\{x; u\}$ could yield a new conservation law of $R\{x; u\}$. Since any point transformation that is a symmetry of PDE system $R\{x; u\}$ leaves invariant the solution manifold of $R\{x; u\}$, it follows that there exist specific functions $A^{\sigma}_{\tau}[W]$ so that $S^{\sigma}[W] = R^{\sigma}[U]$ is of the form

$$S^{\sigma}[W] = R^{\sigma}[U] = A^{\sigma}_{\tau}[W]R^{\tau}[W]. \tag{48}$$

Hence through formula (45) one obtains a symmetry mapping formula for conservation laws.

Corollary 3.16. *If the invertible point transformation $(x, u) \longrightarrow (\tilde{x}(x, u), \tilde{u}(x, u))$ is a symmetry of the PDE system $R\{x; u\}$, then a conservation law $D_i\phi^i[u] = 0$ of $R\{x; u\}$ yields the conservation law*

$$D_i\psi^i[u] = 0, \tag{49}$$

of $R\{x; u\}$ with fluxes given by

$$\psi^1[u] = \begin{pmatrix} \phi^1[\tilde{u}] & D_2\tilde{x}^1 & \dots & D_n\tilde{x}^1 \\ \phi^2[\tilde{u}] & D_2\tilde{x}^2 & \dots & D_n\tilde{x}^2 \\ \vdots & \vdots & \ddots & \vdots \\ \phi^n[\tilde{u}] & D_2\tilde{x}^n & \dots & D_n\tilde{x}^n \end{pmatrix}_{n\times n}, \dots,$$

$$\psi^n[u] = \begin{pmatrix} D_1\tilde{x}^1 & \dots & D_{n-1}\tilde{x}^1 & \phi^1[\tilde{u}] \\ D_1\tilde{x}^2 & \dots & D_{n-1}\tilde{x}^2 & \phi^2[\tilde{u}] \\ \vdots & \vdots & \ddots & \vdots \\ D_1\tilde{x}^n & \dots & D_{n-1}\tilde{x}^n & \phi^n[\tilde{u}] \end{pmatrix}_{n\times n}. \tag{50}$$

Theorem 3.17. *Suppose the point transformation (43) is a symmetry of PDE system (7). If $\{\Lambda_\sigma[U]\}_{\sigma=1}^N$ is a set of multipliers for a conservation law of $R\{x; u\}$ with fluxes $\phi^i[u]$, then*

$$\hat{\Lambda}_\tau[W]R^{\tau}[W] = \tilde{D}_i\psi^i[W], \tag{51}$$

where

$$\hat{\Lambda}_\tau[W] = J[W]A^{\sigma}_{\tau}[W]\Lambda[U(z, W)], \qquad \tau = 1, \dots, N \tag{52}$$

with $U(z, W)$ (and its derivatives) given by the transformation (43) (and its natural extensions). In (51), $\psi^i[W]$ is given by (50) and, in (52), the Jacobian determinant $J[W]$ and $A^{\sigma}_{\tau}[W]$ are given by (46) and (52), respectively.

Corollary 3.18. *If $\{\Lambda_\sigma[U]\}_{\sigma=1}^N$ is a set of multipliers for a conservation law of PDE system $R\{x; u\}$ (7) and PDE system $R\{x; u\}$ is invariant under the point transformation $(x, u) \longrightarrow (\tilde{x}(x, u), \tilde{u}(x, u))$, then $\{\hat{\Lambda}_\tau[U]\}_{\tau=1}^N$ yields a set of multipliers for a conservation law of $R\{x; u\}$ where $\hat{\Lambda}_\tau[U] = J[\tilde{U}]A^{\sigma}_{\tau}[\tilde{U}]\Lambda_\sigma[U]$.*

Proposition 3.19. *A set of multipliers $\{\hat{\Lambda}_\tau[U]\}_{\tau=1}^N$ yields a new conservation law of PDE system $R\{x; u\}$(7) if and only if this set is independent of $\{\Lambda_\sigma[U]\}_{\sigma=1}^N$ on all solutions $U(x) = u(x)$ of $R\{x; u\}$, i.e., $\hat{\Lambda}_\tau[U] \neq c\Lambda_\tau[U] \quad \tau = 1, \dots, N$, for any constant c.*

Example 3.20. We have already shown that the wave equation has conservation laws with fluxes given by Table 1 and the set of multipliers (18).

(I) Consider the fourth point symmetry with transformation

$$\left(\tilde{t} = t + \varepsilon, \tilde{x} = x, \tilde{y} = y, \tilde{z} = z, \tilde{u} = u\right). \tag{53}$$

Under this action, one has

$$J[W] = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 1,$$

and

$$u_{tt} = \tilde{u}_{\tilde{t}\tilde{t}}, \qquad u_{xx} = \tilde{u}_{\tilde{x}\tilde{x}}, \qquad u_{yy} = \tilde{u}_{\tilde{y}\tilde{y}}, \qquad u_{zz} = \tilde{u}_{\tilde{z}\tilde{z}}.$$

So we have

$$S[W] = R[U] = u_{tt} - u_{xx} - u_{yy} - u_{zz}. \tag{54}$$

By utilizing Corollary 3.18, one gets a new set of multipliers

$$\hat{\Lambda}_1 = J[W] A_1^{\sigma} \Lambda_{\sigma} = \begin{cases} \tilde{ut} & \sigma = 2 \\ \tilde{ux} & \sigma = 3 \\ \tilde{uy} & \sigma = 4 \\ \tilde{uz} & \sigma = 5 \\ \tilde{t}\tilde{ux} + \tilde{x}\tilde{ut} & \sigma = 6 \\ \tilde{t}\tilde{uy} + \tilde{y}\tilde{ut} + \varepsilon(-\tilde{uy}) & \sigma = 7 \\ \tilde{y}\tilde{ux} - \tilde{x}\tilde{uy} & \sigma = 8 \\ \tilde{t}\tilde{uz} + \tilde{z}\tilde{ut} + \varepsilon(-\tilde{uz}) & \sigma = 9. \end{cases}$$

The conservation laws associated with the set of new multipliers $\hat{\Lambda}_1$ digest in Table 5.

(II) Now consider the sixth point symmetry

$$\left(\tilde{t} = t, \tilde{x} = x, \tilde{y} = z\sin(\varepsilon) + y\cos(\varepsilon), \tilde{z} = z\cos(\varepsilon) - y\sin(\varepsilon), \tilde{u} = u\right). \tag{55}$$

Hence

$$J[W] = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\varepsilon & \sin\varepsilon \\ 0 & 0 & \sin\varepsilon & \cos\varepsilon \end{vmatrix} = 1.$$

In the sequence of calculation we have

$$u_{tt} = \tilde{u}_{\tilde{t}\tilde{t}}, \qquad u_{yy} = \cos^2\varepsilon\tilde{u}_{\tilde{y}\tilde{y}} + \sin^2\varepsilon\tilde{u}_{\tilde{z}\tilde{z}} - \sin 2\varepsilon\tilde{u}_{\tilde{y}\tilde{z}},$$

$$u_{xx} = \tilde{u}_{\tilde{x}\tilde{x}}, \qquad u_{zz} = \sin^2\varepsilon\tilde{u}_{\tilde{y}\tilde{y}} + \cos^2\varepsilon\tilde{u}_{\tilde{z}\tilde{z}} + \sin 2\varepsilon\tilde{u}_{\tilde{y}\tilde{z}}.$$

Thus,

$$S[W] = R[U] = u_{tt} - u_{xx} - u_{yy} - u_{zz}. \tag{56}$$

After applying Corollary 3.18, one gets a new set of multipliers

$$\hat{\Lambda}_1 = J[W]A_1^\sigma \Lambda_\sigma = \begin{cases} \tilde{u_t} & \sigma = 2 \\ \tilde{u_x} & \sigma = 3 \\ \tilde{u_y}\cos\varepsilon - \tilde{u_z}\sin\varepsilon & \sigma = 4 \\ \tilde{u_y}\sin\varepsilon + \tilde{u_z}\cos\varepsilon & \sigma = 5 \\ \tilde{t}\tilde{u_x} + \tilde{x}\tilde{u_t} & \sigma = 6 \\ \tilde{t}\left(\tilde{u_y}\cos\varepsilon - \tilde{u_z}\sin\varepsilon\right) + \tilde{u_t}\,(\tilde{y}\cos\varepsilon - \tilde{z}\sin\varepsilon) & \sigma = 7 \\ (\tilde{y}\tilde{u_x} - \tilde{x}\tilde{u_y})\cos\varepsilon + (\tilde{x}\tilde{u_z} - \tilde{z}\tilde{u_x})\sin\varepsilon & \sigma = 8 \\ (\tilde{t}\tilde{u_y} + \tilde{y}\tilde{u_t})\sin\varepsilon + (\tilde{t}\tilde{u_z} + \tilde{z}\tilde{u_t})\cos\varepsilon & \sigma = 9. \end{cases}$$

That conservation laws computed with new multipliers $\hat{\Lambda}_1$, are summarized in Table 6.

4. Applications of conservation laws for constructing solutions of PDEs

Let us assume that the system (7), has a conservation law

$$[D_i(C^i) = 0]|_{(7)} = 0, \tag{57}$$

where $C^i = C^i(x, u, u_{(1)}, \ldots)$. $\quad i = 1, \ldots, n$.

We write the conservation Eq. (57) in the form

$$D_i(C^i) = \mu^\alpha F_\alpha(x, u, u_{(1)}, \ldots, u_{(s)}) \tag{58}$$

for a given conserved vector $C = (C^1, \ldots, C^n)$. The coefficients μ^α in Eq. (58) are known functional $\mu^\alpha = \mu^\alpha(x, u, u_{(1)}, \ldots)$.

We will construct particular solutions of the system (7) by requiring that on these solutions the vector C reduces to the following conserved vector:

$$C = \left(C^1(x^2, \ldots, x^n), \ldots, C^n(x^1, \ldots, x^{n-1})\right) \tag{59}$$

with

$$\begin{aligned} &C^1(x, u, u_{(1)}, \ldots) = h^1(x^2, x^3, \ldots, x^n), \\ &C^2(x, u, u_{(1)}, \ldots) = h^2(x^1, x^3, \ldots, x^n), \qquad \ldots \\ &C^n(x, u, u_{(1)}, \ldots) = h^n(x^1, x^2, \ldots, x^{n-1}). \end{aligned} \tag{60}$$

The differential constraints (60) can be equivalently written as follows:

$$\begin{aligned} &D_1[C^1(x, u, u_{(1)}, \ldots)] = 0, \\ &D_2[C^1(x, u, u_{(1)}, \ldots)] = 0, \qquad \ldots \\ &D_n[C^1(x, u, u_{(1)}, \ldots)] = 0. \end{aligned} \tag{61}$$

Example 4.1. Let us apply the method to Eq. (2). We will construct a particular solution of Eq. (2) using the simplest conservation law:

$$D_t(u_t) + D_x(-u_x) + D_y(-u_y) + D_z(-u_z) = 0. \tag{62}$$

Table 5
Multi-row table.

$\sigma = 2$	ψ^t	$\frac{1}{2}(\tilde{u}_{\tilde{t}}^2 + \tilde{u}_{\tilde{x}}^2 + \tilde{u}_{\tilde{y}}^2 + \tilde{u}_{\tilde{z}}^2)$	0
	ϕ^x	$-\tilde{u}_{\tilde{x}}\tilde{u}_{\tilde{t}}$	0
	ϕ^y	$-\tilde{u}_{\tilde{y}}\tilde{u}_{\tilde{t}}$	0
	ϕ^z	$-\tilde{u}_{\tilde{z}}\tilde{u}_{\tilde{z}}$	0
$\sigma = 3$	ψ^t	$\tilde{u}_{\tilde{x}}\tilde{u}_{\tilde{t}}$	0
	ϕ^x	$\frac{1}{2}(\tilde{u}_{\tilde{y}}^2 + \tilde{u}_{\tilde{z}}^2 - \tilde{u}_{\tilde{x}}^2 - \tilde{u}_{\tilde{t}}^2)$	0
	ϕ^y	$-\tilde{u}_{\tilde{x}}\tilde{u}_{\tilde{y}}$	0
	ϕ^z	$-\tilde{u}_{\tilde{x}}\tilde{u}_{\tilde{z}}$	0
$\sigma = 4$	ψ^t	$\tilde{u}_{\tilde{y}}\tilde{u}_{\tilde{t}}$	0
	ϕ^x	$-\tilde{u}_{\tilde{x}}\tilde{u}_{\tilde{y}}$	0
	ϕ^y	$\frac{1}{2}(\tilde{u}_{\tilde{x}}^2 + \tilde{u}_{\tilde{z}}^2 - \tilde{u}_{\tilde{y}}^2 - \tilde{u}_{\tilde{t}}^2)$	0
	ϕ^z	$-\tilde{u}_{\tilde{y}}\tilde{u}_{\tilde{z}}$	0
$\sigma = 5$	ψ^t	$\tilde{u}_{\tilde{z}}\tilde{u}_{\tilde{t}}$	0
	ϕ^x	$-\tilde{u}_{\tilde{x}}\tilde{u}_{\tilde{z}}$	0
	ϕ^y	$-\tilde{u}_{\tilde{y}}\tilde{u}_{\tilde{z}}$	0
	ϕ^z	$\frac{1}{2}(\tilde{u}_{\tilde{x}}^2 + \tilde{u}_{\tilde{y}}^2 - \tilde{u}_{\tilde{z}}^2 - \tilde{u}_{\tilde{t}}^2)$	0
$\sigma = 6$	ψ^t	$\frac{1}{2}\tilde{x}(\tilde{u}_{\tilde{t}}^2 + \tilde{u}_{\tilde{x}}^2 + \tilde{u}_{\tilde{y}}^2 + \tilde{u}_{\tilde{z}}^2) + \tilde{t}\tilde{u}_{\tilde{x}}\tilde{u}_{\tilde{t}}$	$-\varepsilon(\tilde{u}_{\tilde{x}}\tilde{u}_{\tilde{t}})$
	ϕ^x	$\frac{1}{2}\tilde{t}(\tilde{u}_{\tilde{y}}^2 + \tilde{u}_{\tilde{z}}^2 - \tilde{u}_{\tilde{x}}^2 - \tilde{u}_{\tilde{t}}^2) - \tilde{x}\tilde{u}_{\tilde{x}}\tilde{u}_{\tilde{t}}$	$\frac{1}{2}\varepsilon(\tilde{u}_{\tilde{x}}^2 + \tilde{u}_{\tilde{t}}^2 - \tilde{u}_{\tilde{y}}^2 - \tilde{u}_{\tilde{z}}^2)$
	ϕ^y	$-(\tilde{x}\tilde{u}_{\tilde{y}}\tilde{u}_{\tilde{t}} + \tilde{t}\tilde{u}_{\tilde{x}}\tilde{u}_{\tilde{y}})$	$\varepsilon(\tilde{u}_{\tilde{x}}\tilde{u}_{\tilde{y}})$
	ϕ^z	$-(\tilde{x}\tilde{u}_{\tilde{z}}\tilde{u}_{\tilde{t}} + \tilde{t}\tilde{u}_{\tilde{x}}\tilde{u}_{\tilde{z}})$	$\varepsilon(\tilde{u}_{\tilde{x}}\tilde{u}_{\tilde{z}})$
$\sigma = 7$	ψ^t	$\frac{1}{2}\tilde{y}(\tilde{u}_{\tilde{t}}^2 + \tilde{u}_{\tilde{x}}^2 + \tilde{u}_{\tilde{y}}^2 + \tilde{u}_{\tilde{z}}^2) + \tilde{t}\tilde{u}_{\tilde{y}}\tilde{u}_{\tilde{t}}$	$-\tilde{u}_{\tilde{y}}\tilde{u}_{\tilde{t}}$
	ϕ^x	$-\tilde{u}_{\tilde{x}}(\tilde{t}\tilde{u}_{\tilde{y}} + \tilde{y}\tilde{u}_{\tilde{t}})$	$\tilde{u}_{\tilde{x}}\tilde{u}_{\tilde{y}}$
	ϕ^y	$-\frac{1}{2}\tilde{t}(\tilde{u}_{\tilde{y}}^2 + \tilde{u}_{\tilde{t}}^2 - \tilde{u}_{\tilde{x}}^2 - \tilde{u}_{\tilde{z}}^2) - \tilde{y}\tilde{u}_{\tilde{y}}\tilde{u}_{\tilde{t}}$	$\frac{1}{2}(\tilde{u}_{\tilde{y}}^2 + \tilde{u}_{\tilde{t}}^2 - \tilde{u}_{\tilde{x}}^2 - \tilde{u}_{\tilde{z}}^2)$
	ϕ^z	$-\tilde{u}_{\tilde{z}}(\tilde{t}\tilde{u}_{\tilde{y}} + \tilde{y}\tilde{u}_{\tilde{t}})$	$\tilde{u}_{\tilde{y}}\tilde{u}_{\tilde{z}}$
$\sigma = 8$	ψ^t	$(\tilde{y}\tilde{u}_{\tilde{x}}\tilde{u}_{\tilde{t}} - \tilde{x}\tilde{u}_{\tilde{y}}\tilde{u}_{\tilde{t}})$	0
	ϕ^x	$\frac{1}{2}\tilde{y}(\tilde{u}_{\tilde{y}}^2 + \tilde{u}_{\tilde{z}}^2 - \tilde{u}_{\tilde{x}}^2 - \tilde{u}_{\tilde{t}}^2) + \tilde{x}\tilde{u}_{\tilde{x}}\tilde{u}_{\tilde{y}}$	0
	ϕ^y	$\frac{1}{2}\tilde{x}(\tilde{u}_{\tilde{y}}^2 + \tilde{u}_{\tilde{t}}^2 - \tilde{u}_{\tilde{x}}^2 - \tilde{u}_{\tilde{z}}^2) - \tilde{y}\tilde{u}_{\tilde{x}}\tilde{u}_{\tilde{y}}$	0
	ϕ^z	$(\tilde{x}\tilde{u}_{\tilde{y}}\tilde{u}_{\tilde{z}} - \tilde{y}\tilde{u}_{\tilde{x}}\tilde{u}_{\tilde{z}})$	0
$\sigma = 9$	ψ^t	$\frac{1}{2}\tilde{z}(\tilde{u}_{\tilde{t}}^2 + \tilde{u}_{\tilde{x}}^2 + \tilde{u}_{\tilde{y}}^2 + \tilde{u}_{\tilde{z}}^2) + \tilde{t}\tilde{u}_{\tilde{z}}\tilde{u}_{\tilde{t}}$	$-\tilde{u}_{\tilde{z}}\tilde{u}_{\tilde{t}}$
	ϕ^x	$-\tilde{u}_{\tilde{x}}(\tilde{t}\tilde{u}_{\tilde{z}} + \tilde{z}\tilde{u}_{\tilde{t}})$	$\tilde{u}_{\tilde{x}}\tilde{u}_{\tilde{z}}$
	ϕ^y	$-\tilde{u}_{\tilde{y}}(\tilde{t}\tilde{u}_{\tilde{z}} + \tilde{z}\tilde{u}_{\tilde{t}})$	$\tilde{u}_{\tilde{y}}\tilde{u}_{\tilde{z}}$
	ϕ^z	$-\frac{1}{2}\tilde{t}(\tilde{u}_{\tilde{t}}^2 + \tilde{u}_{\tilde{z}}^2 - \tilde{u}_{\tilde{x}}^2 - \tilde{u}_{\tilde{y}}^2) - \tilde{z}\tilde{u}_{\tilde{z}}\tilde{u}_{\tilde{t}}$	$-\frac{1}{2}(\tilde{u}_{\tilde{x}}^2 + \tilde{u}_{\tilde{y}}^2 - \tilde{u}_{\tilde{z}}^2 - \tilde{u}_{\tilde{t}}^2)$

By use of integration of Eq. (60) we have

$$u = h^1(x, y, z)t - h^2(t, y, z)x - h^3(t, x, z)y - h^4(t, x, y)z. \tag{63}$$

Inserting (63) in Eq. (2), we find the following four reduced equations:

$$\begin{aligned} h^1_{xx} + h^1_{yy} + h^1_{zz} &= 0, \\ h^2_{tt} - h^2_{yy} - h^2_{zz} &= 0, \\ h^3_{tt} - h^3_{xx} - h^3_{zz} &= 0, \\ h^4_{tt} - h^4_{xx} - h^4_{yy} &= 0. \end{aligned} \tag{64}$$

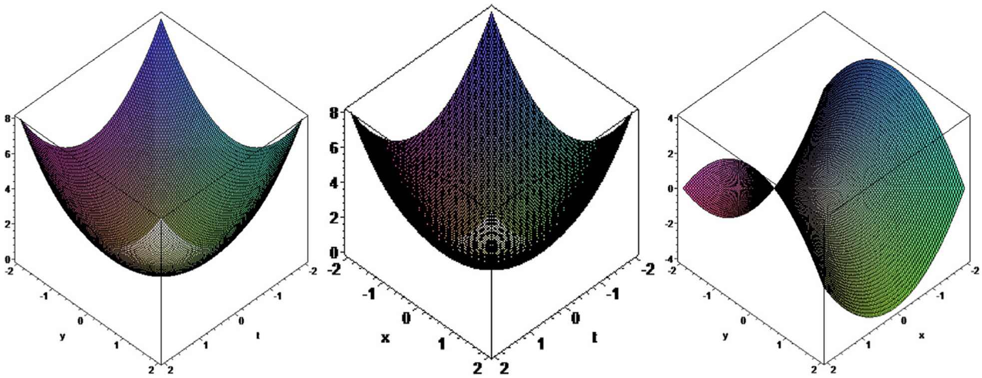
Table 6
Conservation laws.

$\sigma=2$	ψ^t	$\frac{1}{2}(\tilde{u}_{\tilde{t}}^2+\tilde{u}_{\tilde{x}}^2+\tilde{u}_{\tilde{y}}^2+\tilde{u}_{\tilde{z}}^2)$
	ϕ^x	$-\tilde{u}_{\tilde{x}}\tilde{u}_{\tilde{t}}$
	ϕ^y	$-\tilde{u}_{\tilde{y}}\tilde{u}_{\tilde{t}}$
	ϕ^z	$-\tilde{u}_{\tilde{z}}\tilde{u}_{\tilde{z}}$
$\sigma=3$	ψ^t	$\tilde{u}_{\tilde{x}}\tilde{u}_{\tilde{t}}$
	ϕ^x	$\frac{1}{2}(\tilde{u}_{\tilde{y}}^2+\tilde{u}_{\tilde{z}}^2-\tilde{u}_{\tilde{x}}^2-\tilde{u}_{\tilde{t}}^2)$
	ϕ^y	$-\tilde{u}_{\tilde{x}}\tilde{u}_{\tilde{y}}$
	ϕ^z	$-\tilde{u}_{\tilde{x}}\tilde{u}_{\tilde{z}}$
$\sigma=4$	ψ^t	$\tilde{u}_{\tilde{t}}(u\tilde{y}\cos\varepsilon-u\tilde{z}\sin\varepsilon)$
	ϕ^x	$-\tilde{u}_{\tilde{x}}(u\tilde{y}\cos\varepsilon-u\tilde{z}\sin\varepsilon)$
	ϕ^y	$\frac{1}{2}\left(\tilde{u}_{\tilde{x}}^2-\tilde{u}_{\tilde{t}}^2+\tilde{u}_{\tilde{y}}^2(\sin^2\varepsilon-\cos^2\varepsilon)-\tilde{u}_{\tilde{z}}^2(\sin^2\varepsilon-\cos^2\varepsilon)+4\sin\varepsilon\cos\varepsilon\tilde{u}_{\tilde{y}}\tilde{u}_{\tilde{z}}\right)$
	ϕ^z	$\frac{1}{2}\sin 2\varepsilon(\tilde{u}_{\tilde{z}}^2-\tilde{u}_{\tilde{y}}^2)+u\tilde{y}u\tilde{z}(\sin^2\varepsilon-\cos^2\varepsilon)$
$\sigma=5$	ψ^t	$\tilde{u}_{\tilde{t}}(\sin\varepsilon\tilde{u}_{\tilde{y}}+\cos\varepsilon\tilde{u}_{\tilde{z}})$
	ϕ^x	$-\tilde{u}_{\tilde{x}}(\sin\varepsilon\tilde{u}_{\tilde{y}}+\cos\varepsilon\tilde{u}_{\tilde{z}})$
	ϕ^y	$\frac{1}{2}\sin 2\varepsilon(\tilde{u}_{\tilde{z}}^2-\tilde{u}_{\tilde{y}}^2)+u\tilde{y}u\tilde{z}(\sin^2\varepsilon-\cos^2\varepsilon)$
	ϕ^z	$\frac{1}{2}\left(\tilde{u}_{\tilde{x}}^2-\tilde{u}_{\tilde{t}}^2-\tilde{u}_{\tilde{y}}^2(\sin^2\varepsilon-\cos^2\varepsilon)+\tilde{u}_{\tilde{z}}^2(\sin^2\varepsilon-\cos^2\varepsilon)-4\sin\varepsilon\cos\varepsilon\tilde{u}_{\tilde{y}}\tilde{u}_{\tilde{z}}\right)$
$\sigma=6$	ψ^t	$\frac{1}{2}\tilde{x}(\tilde{u}_{\tilde{t}}^2+\tilde{u}_{\tilde{x}}^2+\tilde{u}_{\tilde{y}}^2+\tilde{u}_{\tilde{z}}^2)+\tilde{t}\tilde{u}_{\tilde{x}}\tilde{u}_{\tilde{t}}$
	ϕ^x	$\frac{1}{2}\tilde{t}(\tilde{u}_{\tilde{y}}^2+\tilde{u}_{\tilde{z}}^2-\tilde{u}_{\tilde{x}}^2-\tilde{u}_{\tilde{t}}^2)-\tilde{x}\tilde{u}_{\tilde{x}}\tilde{u}_{\tilde{t}}$
	ϕ^y	$-(\tilde{x}\tilde{u}_{\tilde{y}}\tilde{u}_{\tilde{t}}+\tilde{t}\tilde{u}_{\tilde{x}}\tilde{u}_{\tilde{y}})$
	ϕ^z	$-(\tilde{x}\tilde{u}_{\tilde{z}}\tilde{u}_{\tilde{t}}+\tilde{t}\tilde{u}_{\tilde{x}}\tilde{u}_{\tilde{z}})$
$\sigma=7$	ψ^t	$\frac{1}{2}(\tilde{y}\cos\varepsilon-\tilde{z}\sin\varepsilon)(\tilde{u}_{\tilde{t}}^2+\tilde{u}_{\tilde{x}}^2+\tilde{u}_{\tilde{y}}^2+\tilde{u}_{\tilde{z}}^2)+\tilde{t}(\tilde{u}_{\tilde{y}}\cos\varepsilon-\tilde{u}_{\tilde{z}}\sin\varepsilon)$
	ϕ^x	$-\tilde{u}_{\tilde{x}}\left(\cos\varepsilon(\tilde{t}\tilde{u}_{\tilde{y}}+\tilde{y}\tilde{u}_{\tilde{t}})-\sin\varepsilon(\tilde{t}\tilde{u}_{\tilde{z}}+\tilde{z}\tilde{u}_{\tilde{t}})\right)$
	ϕ^y	$\frac{1}{2}\tilde{t}\left(\tilde{u}_{\tilde{t}}^2-\tilde{u}_{\tilde{x}}^2-\tilde{u}_{\tilde{y}}^2(\sin^2\varepsilon-\cos^2\varepsilon)+\tilde{u}_{\tilde{z}}^2(\sin^2\varepsilon-\cos^2\varepsilon)-4\sin\varepsilon\cos\varepsilon\tilde{u}_{\tilde{y}}\tilde{u}_{\tilde{z}}\right)$
	ϕ^z	$-(\tilde{u}_{\tilde{y}}\cos\varepsilon+\tilde{u}_{\tilde{z}}\sin\varepsilon)\left(\cos\varepsilon(\tilde{t}\tilde{u}_{\tilde{y}}+\tilde{y}\tilde{u}_{\tilde{t}})-\sin\varepsilon(\tilde{t}\tilde{u}_{\tilde{z}}+\tilde{z}\tilde{u}_{\tilde{t}})\right)$
$\sigma=8$	ψ^t	$\tilde{u}_{\tilde{t}}\left(\cos\varepsilon(\tilde{y}\tilde{u}_{\tilde{x}}-\tilde{x}\tilde{u}_{\tilde{y}})+\sin\varepsilon(\tilde{x}\tilde{u}_{\tilde{z}}-\tilde{z}\tilde{u}_{\tilde{x}})\right)$
	ϕ^x	$\frac{1}{2}(\tilde{y}\cos\varepsilon-\tilde{z}\sin\varepsilon)(\tilde{u}_{\tilde{t}}^2+\tilde{u}_{\tilde{x}}^2+\tilde{u}_{\tilde{y}}^2+\tilde{u}_{\tilde{z}}^2)+\tilde{t}\tilde{u}_{\tilde{x}}(\tilde{u}_{\tilde{y}}\cos\varepsilon-\tilde{u}_{\tilde{z}}\sin\varepsilon)$
	ϕ^y	V
	ϕ^z	$-(\tilde{u}_{\tilde{y}}\sin\varepsilon+\tilde{u}_{\tilde{z}}\cos\varepsilon)\left(\cos\varepsilon(\tilde{y}\tilde{u}_{\tilde{x}}-\tilde{x}\tilde{u}_{\tilde{y}})+\sin\varepsilon(\tilde{x}\tilde{u}_{\tilde{z}}-\tilde{z}\tilde{u}_{\tilde{x}})\right)$
$\sigma=9$	ψ^t	$\frac{1}{2}(\tilde{y}\sin\varepsilon+\tilde{z}\cos\varepsilon)(\tilde{u}_{\tilde{t}}^2+\tilde{u}_{\tilde{x}}^2+\tilde{u}_{\tilde{y}}^2+\tilde{u}_{\tilde{z}}^2)+\tilde{t}\tilde{u}_{\tilde{t}}(\tilde{u}_{\tilde{y}}\sin\varepsilon+\tilde{u}_{\tilde{z}}\cos\varepsilon)$
	ϕ^x	$-\tilde{u}_{\tilde{x}}\left(\sin\varepsilon(\tilde{t}\tilde{u}_{\tilde{y}}+\tilde{y}\tilde{u}_{\tilde{t}})+\cos\varepsilon(\tilde{t}\tilde{u}_{\tilde{z}}+\tilde{z}\tilde{u}_{\tilde{t}})\right)$
	ϕ^y	$-(\cos\varepsilon\tilde{u}_{\tilde{y}}-\sin\varepsilon\tilde{u}_{\tilde{z}})\left(\sin\varepsilon(\tilde{t}\tilde{u}_{\tilde{y}}+\tilde{y}\tilde{u}_{\tilde{t}})+\cos\varepsilon(\tilde{t}\tilde{u}_{\tilde{z}}+\tilde{z}\tilde{u}_{\tilde{t}})\right)$
	ϕ^z	$-\frac{1}{2}\tilde{t}\left(\tilde{u}_{\tilde{t}}^2-\tilde{u}_{\tilde{x}}^2+\tilde{u}_{\tilde{y}}^2(\sin^2\varepsilon-\cos^2\varepsilon)+\tilde{u}_{\tilde{z}}^2(\sin^2\varepsilon-\cos^2\varepsilon)+4\sin\varepsilon\cos\varepsilon\tilde{u}_{\tilde{y}}\tilde{u}_{\tilde{z}}\right)$

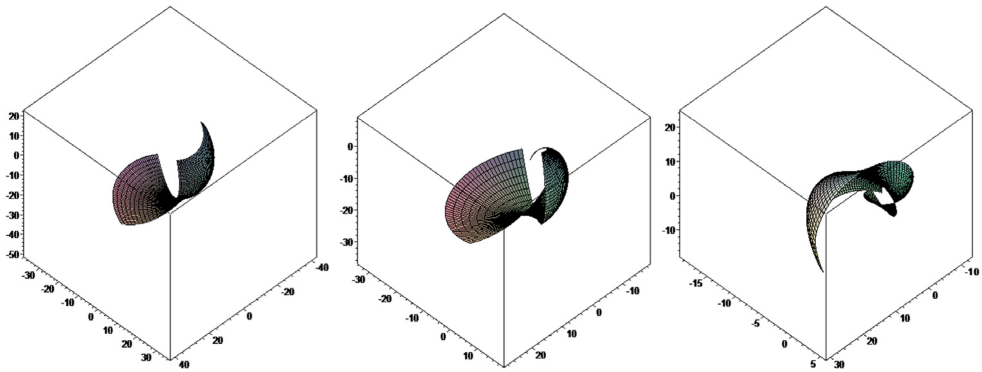
The number of solutions of these reduced equations is infinite, however, we compute some of these solutions via their plot (see Figs. 1 and 2).

5. Conclusion

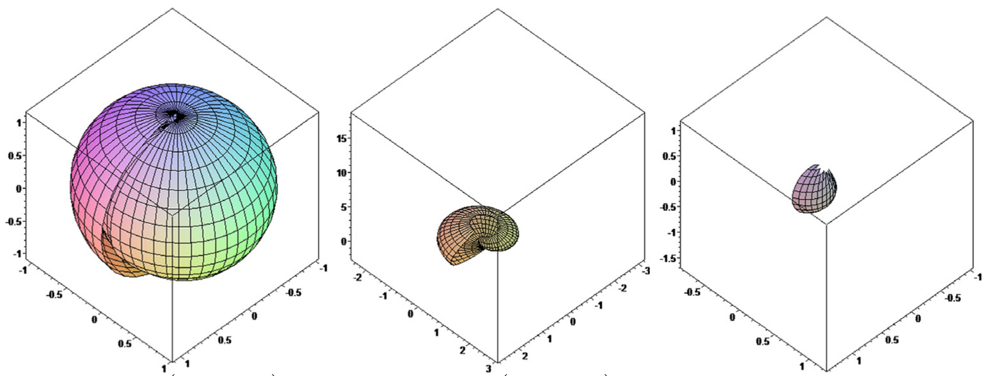
In this paper, we have comprehensively analyzed the problem of symmetries and conservation laws of the wave equation. We have also briefly investigated the structure of the Lie algebra of symmetries from algebraic point of view. The double reduction theory based on symmetry and its associated conserved vector was utilized and many independent exact solutions were obtained. Moreover, the Lie symmetry method was used to derive a solution. Using extended Noether's theorem method and the multipliers method of Anco



(a) $u = t^2 + y^2$. (b) $u = t^2 + x^2$. (c) $u = x^2 - y^2$.

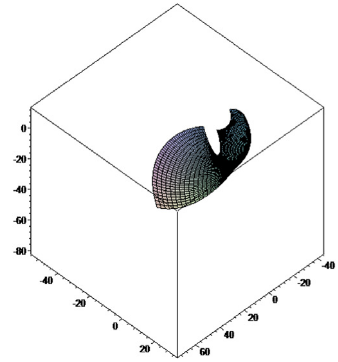


(d) $u = 2t^2 + x^2 + y^2$. (e) $u = t^2 - x^2 + 2y^2$. (f) $u = y^2 - t^2 - 2x^2$.

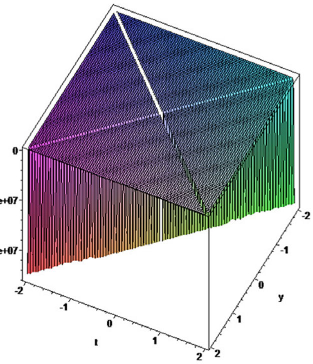


(g) $u = \arctan\left(\sqrt{\frac{x^2+y^2-t^2}{t^2}}\right)$. (h) $u = \operatorname{arctanh}\left(\sqrt{\frac{x^2+y^2-t^2}{x^2}}\right)$. (i) $u = \log\left(\sqrt{\frac{t^2}{x^2+y^2}} + \sqrt{\frac{t^2-x^2-y^2}{x^2+y^2}}\right)$.

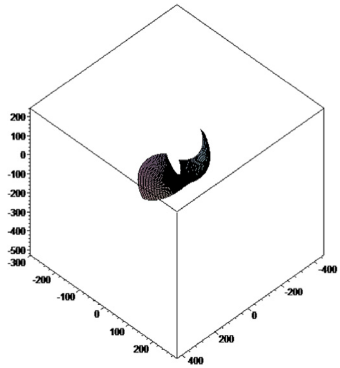
Fig. 1. Pictures of solutions.



(j) $u = y(t^2 + x^2)$.

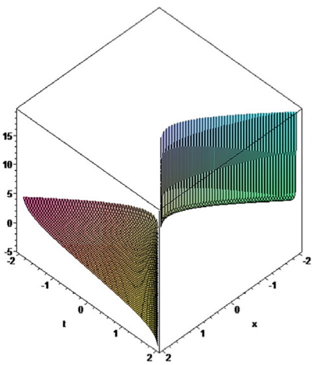


(k) $u = \frac{t}{t^2 - y^2}$.

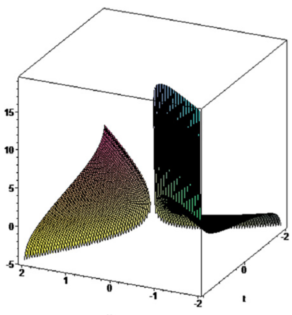


(l) $u = 7tx^2 + 2ty^2 + 3t^3$.

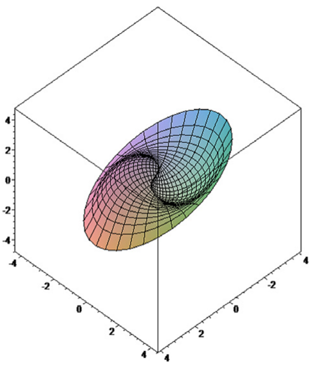
Fig. 1. (*continued*)



(a) $u = \log(\frac{x-t}{x+t})$.



(b) $u = \log(\frac{y-t}{y+t})$.



(c) $u = \log\big(\frac{1}{2} + \frac{x^2}{t^2 - x^2} + \sqrt{\frac{x^4}{(t^2-x^2)^2} + \frac{x^2}{t^2-x^2}}\big)$.

Fig. 2. Pictures of solutions.

and Bluman [1], we constructed some nontrivial conservation laws for this equation. We proved that Eq. (2) is nonlinear self-adjoint and obtain nontrivial conservation laws based on Ibragimov's theorem method and Noether's theorem method that outcomes were summarized in Tables 2, 3, and 4. Finally, we found exact solutions accompanied by their plots. It is noteworthy that in this paper we showed that the whole five methods have some intensity and weakness points. For example we note that there are several limitations to Noether's theorem: It is restricted to variational systems. Thus, we have the difficulty of finding symmetries admitted by the action functional here. Moreover, the use of Noether's theorem to find conservation laws is coordinate-dependent. The extended Noether's method is similar to the Noether's method and some results are the same as the celebrated Noether's theorem. In comparing the direct method with Noether's theorem, it is important to reiterate that conservation laws arise from multipliers for both approaches. But unlike Noether's theorem (also, Ibragimov's theorem), direct method is not limited to PDE systems arising from some variational principle (i.e., self-adjoint PDE systems). None of these complications arises when one computes conservation law multipliers through the direct method. Indeed, the multiplier determining equations are solved off the solution space of the given PDEs. As was observed the major weakness of the direct method is in calculating the fluxes and the densities of the conservation law. Whereas, in this way, either shall be directly finding two polynomials in terms of the dependent variables and their derivatives that be expressed as a divergence expression or is using special formulas which is usually including the integration of multiple sentences. While, in complicated cases which the system of PDEs includes higher-order derivatives, integration is seldom possible. This problem can be resolved by combining Noether's theorem and direct method but as previously mentioned, Noether's theorem has several limitations. Hence it is some times possible. In all these four methods we can find some different conservation laws. In the last method which was based on the transformations arise from the point symmetries, obtained conservation laws are more different from the previous methods. All in all, there are another methods for our purpose such as Hereman–Pool method. It is optimization of direct method that in the case of complicated forms of multipliers and/or equations, for the inversion of divergence operators, one can use homotopy operators.

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