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Iterative approximation of fixed points of contraction mappings in complex valued Banach spaces

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Abstract. We approximate the fixed points of contraction mappings using the Picard–Krasnoselskii hybrid iterative process, which is known to converge faster than all of Picard, Mann and Ishikawa iterations in complex valued Banach spaces. Moreover, we prove analytically and with a numerical example that the Picard–Mann hybrid iteration and the Picard–Krasnoselskii hybrid iteration have the same rate of convergence. Furthermore, we apply our results in finding solutions of delay differential equations in complex valued Banach spaces.

Keywords: Complex valued Banach spaces; Picard–Krasnoselskii hybrid iterative process; Delay differential equations; Picard–Mann hybrid iterative process; Stability; Data dependence

Mathematics Subject Classification: 47H09; 47H10; 49M05; 54H25

1. INTRODUCTION AND PRELIMINARIES

It is known that there is a close relationship between the problem of solving a nonlinear equation and that of approximating fixed points of a corresponding contractive type operator (see, e.g. [6,7,25]). Hence, there are practical and theoretical interests in approximating fixed points of several contractive type operators. We approximate the fixed points of

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contraction mappings using the Picard–Krasnoselskii hybrid iterative process, which is known to converge faster than all of Picard, Mann and Ishikawa iterations in complex valued Banach spaces (see, e.g. [24]). Moreover, we prove analytically and with a numerical example that the Picard–Mann hybrid iteration and the Picard–Krasnoselskii hybrid iteration have the same rate of convergence. Our results generalize and extend several known results in literature, including the results of [3,19,20,23,24] among others. Furthermore, we apply our results in finding the solution of delay differential equations.

Interest in generalized Banach spaces lies in the fact that the metric properties of the problem at hand can be analyzed more accurately. Moreover, convergence domains and estimates on the error distances involved are improved, when compared to the real norm theory (see, e.g. [4,21]). Recently, Argyros et al. [4] presented a weaker convergence analysis of Newton's method than in Traub [33], Meyer [21] among others on a generalized Banach space setting to approximate a locally unique zero of an operator. Their results extend the applicability of Newton's method.

The notion of complex valued metric spaces was introduced by Azam et al. [5] in 2011. They established some fixed point theorems for a pair of mappings satisfying rational inequality. Their results are intended to define rational expressions which are meaningless in cone metric spaces, hence results in this direction cannot be generalized to cone metric spaces, but to complex valued metric spaces. It is known that complex valued metric space is useful in many branches of Mathematics, including number theory, algebraic geometry, applied Mathematics as well as in physics including hydrodynamics, mechanical engineering, thermodynamics and electrical engineering (see, e.g. [29]). Several authors have obtained interesting and applicable results in complex valued metric spaces (see, e.g. [1,2,5,17,27–30]). Since, the introduction of the notion of complex valued metric spaces by Azam et al. [5] in 2011, most results obtained in literature by many authors are existential in nature (see, e.g. [1,2,5,17,27–30]). Consequently, we are motivated to study the approximation of fixed points of some mappings satisfying certain contractive conditions in complex valued Banach spaces.

Interest in the study of delay differential equations stems from the fact that several models in real life problems involve delay differential equations (see, e.g. [24]). For instance, delay models are common in many branches of biological modeling (see [13]). They have been used for describing several aspects of infectious disease dynamics: primary infection [9], drug therapy [22] and immune response [12], among others. Delays have also appeared in the study of chemostat models [39], circadian rhythms [31], epidemiology [11], the respiratory system [36], tumor growth [37] and neural networks [8]. Statistical analysis of ecological data (see e.g. [34,35]) has shown that there is evidence of delay effects in the population dynamics of many species.

Next, we give the following definitions and notations which will be useful in this research.

Definition 1.1 (*[4], [21]*). A generalized Banach space is a triplet $(x, E, |.|)$ such that
(i) X is a linear space over $\mathbb{R}(\mathbb{C})$.
(ii) $E = (E, K, \|.\|)$ is a partially ordered Banach space, i.e.
(ii$_1$) $(E, \|.\|)$ is a real Banach space,
(ii$_2$) E is partially ordered by a closed convex cone K,
(iii$_3$) The norm $\|.\|$ is monotone on K.

(iii) The operator $|.| : X \to K$ satisfies $|x| = 0$ if and only if $x = 0$, $|\theta x| = |\theta||x|$,

$$|x + y| \leq |x| + |y| \quad \text{for each } \ x, y \in X, \ \theta \in \mathbb{R}(\mathbb{C}).$$

(iv) X is a Banach space with respect to the induced norm $\|.\|_i := \|.\|.|.|$.

Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows:

$$z_1 \precsim z_2 \ \text{ if and only if } \ Re(z_1) \leq Re(z_2), \quad Im(z_1) \leq Im(z_2).$$

It follows that

$$z_1 \precsim z_2$$

if one of the following conditions is satisfied:
(i) $Re(z_1) = Re(z_2), \quad Im(z_1) < Im(z_2)$,
(ii) $Re(z_1) < Re(z_2), \quad Im(z_1) = Im(z_2)$,
(iii) $Re(z_1) < Re(z_2), \quad Im(z_1) < Im(z_2)$,
(iv) $Re(z_1) = Re(z_2), \quad Im(z_1) = Im(z_2)$.
In particular, we will write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of (i), (ii), and (iii) is satisfied and we will write $z_1 \prec z_2$ if only (iii) is satisfied. Note that

$$0 \precsim z_1 \precnsim z_2 \Longrightarrow |z_1| < |z_2|,$$

$$z_1 \preceq z_2, \ z_2 \prec z_3 \Longrightarrow z_1 \prec z_3.$$

Definition 1.2 ([5]). Let X be a nonempty set. Suppose that the mapping $d : X \times X \to \mathbb{C}$, satisfies:
1. $0 \precsim d(x, y)$, for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$ for all $x, y \in X$;
3. $d(x, y) \precsim d(x, z) + d(z, y)$, for all $x, y, z \in X$.

Then d is called a complex valued metric on X, and (X, d) is called a complex valued metric space.

Motivated by the results above, we now define a complex valued Banach space as follows:

Definition 1.3. Let E be a linear space over a field $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ (the set of real numbers) or $\mathbb{C}$ (the set of complex numbers). A complex valued norm on E is a complex valued function $\|.\| : E \to \mathbb{C}$ satisfying the following conditions:
1. $\|x\| = 0$ if and only if $x = 0, \ x \in E$;
2. $\|kx\| = |k|.\|x\|$ for all $k \in \mathbb{K}, \ x \in E$;
3. $\|x + y\| \precsim \|x\| + \|y\|$ for all $x, y \in E$.

A linear space with a complex valued norm defined on it is called a *complex valued normed linear space,* denoted by $(E, \|.\|)$. A point $x \in E$ is called an *interior point* of a set $A \subseteq E$ if there exist $0 \prec r \in \mathbb{C}$ such that

$$B(x, r) = \{y \in E : \|x - y\| \prec r\} \subseteq A.$$

A point $x \in E$ is called a limit point of the set A whenever for each $0 \prec r \in \mathbb{C}$, we have

$$B(x, r) \cap (A \setminus E) \neq \emptyset.$$

The set A is said to be open if each element of A is an interior point of A. A subset $B \subseteq E$ is said to be closed if it contains each of its limit point. The family

$$F = \{B(x, r) : x \in E,\ 0 \prec r\}$$

is a sub-basis for a Hausdorff topology τ on E.

Suppose x_n is a sequence in E and $x \in E$. If for all $c \in \mathbb{C}$, with $0 \prec c$ there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $\|x_n - x_{n+m}\| \prec c$, then $\{x_n\}$ is called a Cauchy sequence in $(E, \|.\|)$. If every Cauchy sequence is convergent in $(E, \|.\|)$, then $(E, \|.\|)$ is called a complex valued Banach space.

We now give the following examples of complex valued normed linear spaces.

Example 1.1. Let $E = \mathbb{C}$ be the set of complex numbers. Define $\|.\| : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ by

$$\|z_1 - z_2\| = |x_1 - x_2| + i|y_1 - y_2| \quad \forall z_1, z_2 \in \mathbb{C},$$

where $z_1 = x_1 + iy_1,\ z_2 = x_2 + iy_2$. Clearly, $(\mathbb{C}, \|.\|)$ is a complex valued normed linear space.

Example 1.2. Let $E = \mathbb{C}$ be the set of complex numbers. Define a mapping $\|.\| : \mathbb{C}\times\mathbb{C} \to \mathbb{C}$ by

$$\|z_1 - z_2\| = e^{ik}|z_1 - z_2|, \quad \forall z_1, z_2 \in \mathbb{C},$$
$$\text{where } k \in [0, \frac{\pi}{2}],\ z_1 = x_1 + iy_1,\ z_2 = x_2 + iy_2.$$

Then $(\mathbb{C}, \|.\|)$ is a complex valued normed linear space.

Example 1.3. Let $(C[a, b], \|.\|_\infty)$ be the space of all continuous complex valued functions on a closed interval $[a, b]$, endowed with the Chebyshev norm

$$\|x - y\|_\infty = \max_{t\in[a,b]} |x(t) - y(t)|e^{ik}, \quad x, y \in C[a, b], \quad k \in [0, \frac{\pi}{2}].$$

Then $(C[a, b], \|.\|_\infty)$ is a complex valued Banach space, since the elements of $C[a, b]$ are continuous functions, and convergence with respect to the Chebyshev norm $\|.\|_\infty$ corresponds to uniform convergence. We can easily show that every Cauchy sequence of continuous functions converges to a continuous function, i.e. an element of the space $C[a, b]$.

Next, we prove Lemmas 1.1 and 1.2 as an analogue of ([5], Lemma 2) and ([5], Lemma 3) respectively in complex valued Banach spaces.

Lemma 1.1. *Let $(E, \|.\|)$ be a complex valued Banach space and let $\{x_n\}$ be a sequence in E. Then $\{x_n\}$ converges to x if and only if $|\|x_n - x\|| \to 0$ as $n \to \infty$.*

Proof. Suppose that x_n converges to x. This means that for arbitrary $\epsilon > 0$ and $0 \prec c \in \mathbb{C}$, there exists a natural number N, such that

$$\|x_n - x\| \prec c \text{ for each } n > N.$$

Without loss of generality, let

$$c = \frac{\epsilon}{\sqrt{2}} + i\frac{\epsilon}{\sqrt{2}}.$$

Hence,

$$|\|x_n - x\|| < |c| = \epsilon \;\text{ for each } n > N.$$

It follows that

$$|\|x_n - x\|| \to 0 \;\text{ as } n \to \infty.$$

Conversely, let $|\|x_n - x\|| \to 0$ as $n \to \infty$. Then given $0 \prec c \in \mathbb{C}$, there exists a real number $\delta > 0$ such that for each $z \in \mathbb{C}$

$$|z| < \delta \Longrightarrow z \prec c.$$

For this $\delta > 0$, there exists a natural number N such that

$$|\|x_n - x\|| < \delta \;\text{ for each } n > N.$$

Therefore, $\|x_n - x\| \prec c$ for each $n > N$. Hence, $\{x_n\}$ converges to x as desired. □

Lemma 1.2. *Let $(E, \|.\|)$ be a complex valued Banach space and let $\{x_n\}$ be a sequence in E. Then $\{x_n\}$ is a Cauchy sequence if and only if $|\|x_n - x_{n+m}\|| \to 0$ as $n \to \infty$.*

Proof. Suppose that $\{x_n\}$ is a Cauchy sequence. This means that for arbitrary $\epsilon > 0$ and $0 \prec c \in \mathbb{C}$, there exists a natural number N, such that

$$\|x_n - x_{n+m}\| \prec c \;\text{ for all } n > N.$$

Without loss of generality, let $c = \frac{\epsilon}{\sqrt{2}} + i\frac{\epsilon}{\sqrt{2}}$, then

$$|\|x_n - x_{n+m}\|| \prec |c| = \epsilon \;\text{ for all } n > N.$$

Hence,

$$|\|x_n - x_{n+m}\|| \to 0 \;\text{ as } n \to \infty.$$

Conversely, let $|\|x_n - x_{n+m}\|| \to 0$ as $n \to \infty$. Given arbitrary $0 \prec c \in \mathbb{C}$, there exists a real number $\delta > 0$, such that for each $z \in \mathbb{C}$, we have

$$|z| < \delta \Longrightarrow z \prec c.$$

For this δ, there exists a natural number N such that

$$|\|x_n - x_{n+m}\|| < \delta \;\text{ for each } n > N.$$

This means that $\|x_n - x_{n+m}\| \prec c$ for all $n > N$. Therefore, $\{x_n\}$ is a Cauchy sequence. □

The Picard iterative process is commonly used to approximate the fixed point of contraction mappings $T : D \subseteq E \to D$ satisfying the following contractive condition

$$\|Tx - Ty\| \precsim \delta\|x - y\|, \quad \delta \in (0, 1), \text{ for all } x, y \in D \subseteq E. \tag{1.1}$$

If $\delta = 1$ in relation (1.1), then T is called a nonexpansive mapping. A point $x \in D$ is called a fixed point of the mapping $T : D \to D$ if $Tx = x$. The set of all the fixed points of T is denoted by $F(T) := \{x \in D : Tx = x\}$.

In 2013, Khan [18] introduced the Picard–Mann hybrid iterative process. The iterative process for one mapping case is given by the sequence $\{m_n\}_{n=1}^{\infty}$.

$$\begin{cases} m_1 = m \in D, \\ m_{n+1} = Tz_n, \\ z_n = (1 - \alpha_n)m_n + \alpha_n Tm_n, \quad n \in \mathbb{N}, \end{cases} \tag{1.2}$$

where $\{\alpha_n\}_{n=1}^{\infty}$ is in (0, 1). Khan [18] proved that this iterative process converges faster than all of Picard, Mann and Ishikawa iterative processes in the sense of Berinde [7] for contractive mappings.

Recently, Okeke and Abbas [24] introduced the Picard–Krasnoselskii hybrid iterative process defined by the sequence $\{x_n\}_{n=1}^{\infty}$ as follows:

$$\begin{cases} x_1 = x \in D, \\ x_{n+1} = Ty_n, \\ y_n = (1-\lambda)x_n + \lambda T x_n, \quad n \in \mathbb{N}, \end{cases} \tag{1.3}$$

where $\lambda \in (0, 1)$. The authors proved that this new hybrid iteration process converges faster than all of Picard, Mann, Krasnoselskii and Ishikawa iterative processes in the sense of Berinde [7]. They also used this iterative process to find the solution of delay differential equations.

Definition 1.4 (*[7]*). Let $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$ be two sequences of positive numbers that converge to a, respectively b. Assume there exists

$$l = \lim_{n\to\infty} \frac{|a_n - a|}{|b_n - b|}. \tag{1.4}$$

1. If $l = 0$, then it is said that the sequence $\{a_n\}_{n=0}^{\infty}$ converges to a faster than the sequence $\{b_n\}_{n=0}^{\infty}$ to b;
2. If $0 < l < \infty$, then we say that the sequences $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ have the same rate of convergence.

Definition 1.5 (*[7]*). Let $T, \tilde{T} : D \to D$ be two operators. We say that $\tilde{T}$ is an approximate operator of T if for all $x \in D$ and for a fixed $\varepsilon > 0$ we have

$$\|Tx - \tilde{T}x\| \le \varepsilon. \tag{1.5}$$

Definition 1.6 (*[14–16]*). Let D be a nonempty convex subset of E and $T : D \to D$ be an operator. Assume that $x_1 \in D$ and $x_{n+1} = f(T, x_n)$ defines an iteration scheme which produces a sequence $\{x_n\}_{n=1}^{\infty} \subset D$. Suppose, furthermore, that $\{x_n\}_{n=1}^{\infty}$ converges strongly to $x^* \in F(T) \neq \emptyset$. Let $\{y_n\}_{n=1}^{\infty}$ be any bounded sequence in D and put $\varepsilon_n = \|y_{n+1} - f(T, y_n)\|$.
(1) The iteration scheme $\{x_n\}_{n=1}^{\infty}$ defined by $x_{n+1} = f(T, x_n)$ is said to be T-stable on D if $\lim_{n\to\infty}\varepsilon_n = 0$ implies that $\lim_{n\to\infty} y_n = x^*$.
(2) The iteration scheme $\{x_n\}_{n=1}^{\infty}$ defined by $x_{n+1} = f(T, x_n)$ is said to be almost T-stable on D if $\sum_{n=1}^{\infty}\varepsilon_n < \infty$ implies that $\lim_{n\to\infty} y_n = x^*$.

It is easy to show that an iteration process $\{x_n\}_{n=1}^{\infty}$ which is T-stable on D is almost T-stable on D. However, the converse is not true (see, e.g. [26]).

Lemma 1.3 (*[7]*). *If δ is a real number such that $0 \le \delta < 1$ and $\{\epsilon_n\}_{n=0}^{\infty}$ is a sequence of positive numbers such that $\lim_{n\to\infty}\epsilon_n = 0$, then for any sequence of positive numbers $\{u_n\}_{n=0}^{\infty}$ satisfying*

$$u_{n+1} \le \delta u_n + \epsilon_n, \quad n = 0, 1, 2, \ldots$$

one has $\lim_{n\to\infty} u_n = 0$.

Lemma 1.4 ([38]). *Let $\{\beta_n\}_{n=0}^{\infty}$ and $\{\rho\}_{n=0}^{\infty}$ be nonnegative real sequences satisfying the following inequality:*

$$\beta_{n+1} \leq (1-\lambda_n)\beta_n + \rho_n,$$

where $\lambda_n \in (0,1)$, for all $n \geq n_0$, $\sum_{n=1}^{\infty}\lambda_n = \infty$, and $\frac{\rho_n}{\lambda_n} \to 0$ as $n \to \infty$. Then $\lim_{n\to\infty}\beta_n = 0$.

Lemma 1.5 ([32]). *Let $\{\beta_n\}_{n=0}^{\infty}$ be a nonnegative sequence for which one assumes there exists $n_0 \in \mathbb{N}$, such that for all $n \geq n_0$ one has satisfied the inequality*

$$\beta_{n+1} \leq (1-\mu_n)\beta_n + \mu_n\gamma_n,$$

where $\mu_n \in (0,1)$, for all $n \in \mathbb{N}$, $\sum_{n=0}^{\infty}\mu_n = \infty$ and $\gamma_n \geq 0$, $\forall \mathbb{N}$. Then the following inequality holds

$$0 \leq \limsup_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \gamma_n.$$

2. Convergence analysis of some iterative processes in complex valued Banach spaces

We begin this section with the following results which shows that the Picard–Mann hybrid iterative process (1.2) has the same rate of convergence as the Picard–Krasnoselskii hybrid iterative process (1.3). We also support our analytical proofs with a numerical example.

Proposition 2.1. *Let D be a nonempty closed convex subset of a complex valued normed space $(E, \|.\|)$ and let $T : D \to D$ be a contraction mapping. Suppose that each of the iterative processes* (1.2) and (1.3) *converges to the same fixed point p of T where $\{\alpha_n\}_{n=0}^{\infty}$ and λ are such that $0 < \alpha \leq \lambda, \alpha_n < 1$ for all $n \in \mathbb{N}$ and for some α. Then the Picard–Krasnoselskii hybrid iterative process (1.3) has the same rate of convergence as the Picard–Mann hybrid iterative process (1.2).*

Proof. Using ([18], Proposition 1), we have

$$\|m_{n+1} - p\| \precsim [\delta(1-(1-\delta)\alpha)]^n \|m_1 - p\|. \tag{2.1}$$

Let

$$a_n = [\delta(1-(1-\delta)\alpha)]^n \|m_1 - p\|. \tag{2.2}$$

Similarly, using ([24], Proposition 2.1), we have

$$\|x_{n+1} - p\| \precsim [\delta(1-(1-\delta)\alpha)]^n \|x_1 - p\|. \tag{2.3}$$

Let

$$b_n = [\delta(1-(1-\delta)\alpha)]^n \|x_1 - p\|. \tag{2.4}$$

Now, we compute the rate of convergence of the Picard–Krasnoselskii hybrid iterative process (1.3) as follows:

$$\begin{aligned} \frac{b_n}{a_n} &= \frac{[\delta(1-(1-\delta)\alpha)]^n \|x_1 - p\|}{[\delta(1-(1-\delta)\alpha)]^n \|m_1 - p\|} \\ &= \frac{\|x_1 - p\|}{\|m_1 - p\|}. \end{aligned} \tag{2.5}$$

Table 2.1
Comparison of the speed of convergence among various iterative processes.

Step	Picard–Krasnoselskii	Picard–Mann
1	5.0000000000000	5.0000000000000
2	2.2512843540734	2.2512843540734
3	2.0240689690982	2.0240689690982
4	2.0023366393861	2.0023366393861
5	2.0002271411589	2.0002271411589
6	2.0000220828647	2.0000220828647
7	2.0000021469423	2.0000021469423
8	2.0000002087305	2.0000002087305
9	2.0000000202932	2.0000000202932
10	2.0000000019730	2.0000000019730
11	2.0000000001918	2.0000000001918
12	2.0000000000186	2.0000000000186
13	2.0000000000018	2.0000000000018
14	2.0000000000002	2.0000000000002
15	2.0000000000000	2.0000000000000
⋮	⋮	⋮

Clearly, from (2.1) $m_1 \neq p$, so that $0 \prec \|m_1 - p\| \prec \infty$. Similarly, from (2.3) $x_1 \neq p$, so that $0 \prec \|x_1 - p\| \prec \infty$. Hence,

$$0 < \lim_{n\to\infty} \frac{|\|x_1 - p\||}{|\|m_1 - p\||} = l < \infty. \tag{2.6}$$

This means that the Picard–Krasnoselskii hybrid iterative process (1.3) has the same rate of convergence as the Picard–Mann hybrid iterative process (1.2). The proof of Proposition 2.1 is completed. □

Next, we give a numerical example as a support of the analytical results of Proposition 2.1.

Example 2.1. Let $E = \mathbb{R}$ and $D = [1, 10]$. Let $T : D \to D$ be an operator defined by $Tx = \sqrt[3]{2x + 4}$ for all $x \in D$. Choose $\alpha_n = \lambda = \frac{1}{2}$ for each $n \in \mathbb{N}$, with the initial value $x_1 = 5$. Clearly, T is a contraction mapping with contractive constant $\delta = \frac{1}{\sqrt[3]{4}}$ and a unique fixed point $p = 2$. Table 2.1 shows that the Picard–Krasnoselskii hybrid iterative process (1.3) has the same rate of convergence as the Picard–Mann hybrid iterative process (1.2).

Remark 2.1. Table 2.1 shows that both the Picard–Krasnoselskii hybrid iterative process (1.3) and the Picard–Mann hybrid iterative process (1.2) converge to the fixed point $p = 2$ of T at iteration step number 15. Hence, the iterative processes (1.2) and (1.3) have the same rate of convergence.

Theorem 2.1. *Let D be a nonempty closed convex subset of a complex valued Banach space $(E, \|.\|)$ and $T : D \to D$ be a contraction mapping satisfying contractive condition (1.1). Let $\{m_n\}$ be an iterative sequence generated by (1.2) with real sequence $\{\alpha_n\}_{n=0}^{\infty}$ in $[0, 1]$ satisfying $\sum_{n=0}^{\infty}\alpha_n = \infty$. Then $\{m_n\}$ converges strongly to a unique fixed point of T.*

Proof. The famous Banach theorem guarantees the existence and uniqueness of the fixed point p. We now show that $x_n \to p$ as $n \to \infty$. Using (1.1) and (1.2), we obtain:

$$\begin{aligned}\|z_n - p\| &= \|(1-\alpha_n)m_n + \alpha_n T m_n - p\| \\ &\precsim (1-\alpha_n)\|m_n - p\| + \alpha_n\|Tm_n - p\| \\ &\precsim (1-\alpha_n)\|m_n - p\| + \alpha_n\delta\|m_n - p\| \\ &= (1-\alpha_n(1-\delta))\|m_n - p\|. \end{aligned} \tag{2.7}$$

Using (1.1), (1.2) and relation (2.7), we have:

$$\begin{aligned}\|m_{n+1} - p\| &= \|Tz_n - p\| \\ &\precsim \delta\|z_n - p\| \\ &\precsim \delta(1-\alpha_n(1-\delta))\|m_n - p\|. \end{aligned} \tag{2.8}$$

Using the fact that $(1-\alpha_n(1-\delta)) < 1$ and $\delta \in (0,1)$, we obtain the following inequalities from (2.8).

$$\begin{cases} \|m_{n+1} - p\| \precsim \delta(1-\alpha_n(1-\delta))\|m_n - p\| \\ \|m_n - p\| \precsim \delta(1-\alpha_{n-1}(1-\delta))\|m_{n-1} - p\| \\ \|m_{n-1} - p\| \precsim \delta(1-\alpha_{n-2}(1-\delta))\|m_{n-2} - p\| \\ \vdots \\ \|m_2 - p\| \precsim \delta(1-\alpha_1(1-\delta))\|m_1 - p\|. \end{cases} \tag{2.9}$$

From relation (2.9), we derive

$$\|m_{n+1} - p\| \precsim \|m_1 - p\|\delta^{n+1}\prod_{k=1}^{n}(1-\alpha_k(1-\delta)), \tag{2.10}$$

where $(1-\alpha_k(1-\delta)) \in (0,1)$, since $\delta \in (0,1)$ and $\alpha_k \in [0,1]$ for all $k \in \mathbb{N}$. It is well-known in classical analysis that $1 - x \leq e^{-x}$ for all $x \in [0,1]$. Using these facts together with relation (2.10), we have

$$\|m_{n+1} - p\| \precsim \frac{\|m_1 - p\|\delta^{n+1}}{e^{(1-\delta)\sum_{k=1}^{n}\alpha_k}}. \tag{2.11}$$

Therefore,

$$\lim_{n\to\infty} |\|m_{n+1} - p\|| \leq \left\{\frac{|\|m_1 - p\|\delta^{n+1}|}{|e^{(1-\delta)\sum_{k=1}^{n}\alpha_k}|}\right\} \longrightarrow 0 \text{ as } n \to \infty. \tag{2.12}$$

This means that $\lim_{n\to\infty}\|m_n - p\| = 0$. That is $m_n \to p$ as $n \to \infty$ as desired. The proof of Theorem 2.1 is completed. □

Theorem 2.2. *Let D be a nonempty closed convex subset of a complex valued Banach space $(E, \|.\|)$. Let $T : D \to D$ be a nonexpansive mapping. Let $\{x_n\}$ be a sequence generated by the Picard–Krasnoselskii hybrid iterative process (1.3). Then*
(i) $\lim_{n\to\infty}\|x_n - p\|$ *exists for all* $p \in F(T)$.
(ii) $\lim_{n\to\infty}|\|x_n - Tx_n\|| = 0$.

Proof. Suppose $p \in F(T)$, by (1.3) we have

$$\begin{aligned} \|y_n - p\| &= \|(1-\lambda)x_n + \lambda T x_n - p\| \\ &\precsim (1-\lambda)\|x_n - p\| + \lambda\|T x_n - p\| \\ &\precsim (1-\lambda)\|x_n - p\| + \lambda\|x_n - p\| \\ &= \|x_n - p\|. \end{aligned} \tag{2.13}$$

Hence,

$$\begin{aligned} \|x_{n+1} - p\| &= \|T y_n - p\| \\ &\precsim \|y_n - p\| \\ &\precsim \|x_n - p\|. \end{aligned} \tag{2.14}$$

This shows that the sequence $\{\|x_n - p\|\}$ is decreasing, hence (i) is proved. Suppose

$$\lim_{n\to\infty} \|x_n - p\| = b. \tag{2.15}$$

We next prove part (ii). To do this, we first prove that $\lim_{n\to\infty}\|y_n - p\| = b$. Using (2.14), i.e. $\|x_{n+1} - p\| \precsim \|x_n - p\|$, we have

$$\liminf_{n\to\infty} \|x_{n+1} - p\| \precsim \liminf_{n\to\infty} \|x_n - p\|, \tag{2.16}$$

so that

$$b \precsim \liminf_{n\to\infty} \|y_n - p\|. \tag{2.17}$$

Relation (2.14) implies that

$$\limsup_{n\to\infty} \|y_n - p\| \precsim b. \tag{2.18}$$

Using (2.17) and (2.18), we have

$$\lim_{n\to\infty} \|y_n - p\| = b. \tag{2.19}$$

Next, $\|T x_n - p\| \precsim \|x_n - p\|$ implies that

$$\limsup_{n\to\infty} \|T x_n - p\| \precsim b. \tag{2.20}$$

Using (2.15), (2.19), (2.20) and Lemma 1.1, we have

$$\lim_{n\to\infty} |\|x_n - T x_n\|| = 0. \tag{2.21}$$

This completes the proof of Theorem 2.2. □

Next, we obtain the following corollary as a consequence of Theorem 2.2.

Corollary 2.3. *Let D be a nonempty closed convex subset of a complex valued Banach space $(E, \|.\|)$. Let $T : D \to D$ be a contraction mapping satisfying contractive condition (1.1). Let $\{x_n\}$ be a sequence generated by the Picard–Krasnoselskii hybrid iterative process (1.3). Then*
(i) $\lim_{n\to\infty}\|x_n - p\|$ *exists for all* $p \in F(T)$.
(ii) $\lim_{n\to\infty}|\|x_n - T x_n\|| = 0$.

Proof. The proof of Corollary 2.3 follows the same line as in Theorem 2.2. □

Next, we prove the following results using a contractive condition satisfying rational expression.

Proposition 2.2. *Let D be a nonempty closed convex subset of a complex valued normed space $(E, \|.\|)$ and let $T : D \to D$ be a mapping defined as follows*

$$\|Tx - Ty\| \precsim \frac{\varphi(\|x - Tx\|) + a\|x - y\|}{1 + M\|x - Tx\|}, \quad \forall x, y \in D, \; a \in [0, 1), \quad M \geq 0, \tag{2.22}$$

where $\varphi : \mathbb{C}_+ \to \mathbb{C}_+$ is a monotone increasing function such that $\varphi(0) = 0$. Suppose that each of the iterative processes (1.2) and (1.3) *converges to the same fixed point p of T where $\{\alpha_n\}_{n=0}^{\infty}$ and λ are such that $0 < \alpha \leq \lambda, \alpha_n < 1$ for all $n \in \mathbb{N}$ and for some α. Then the Picard–Krasnoselskii hybrid iterative process* (1.3) *has the same rate of convergence as the Picard–Mann hybrid iterative process* (1.2).

Proof. Suppose that p is the fixed point of the mapping T. We obtain the following using relations (2.22) and (1.2)

$$\begin{aligned}
\|m_{n+1} - p\| &= \|Tz_n - p\| \\
&\precsim \frac{\varphi(\|p - Tp\|) + a\|z_n - p\|}{1 + M\|p - Tp\|} \\
&= \frac{\varphi(\|0\|) + a\|z_n - p\|}{1 + M\|0\|} \\
&= a\|(1 - \alpha_n)m_n + \alpha_n Tm_n - p\| \\
&\precsim a(1 - \alpha_n)\|m_n - p\| + a\alpha_n\|Tm_n - p\| \\
&\precsim a(1 - \alpha_n)\|m_n - p\| + a\alpha_n\left[\frac{\varphi(\|p - Tp\|) + a\|m_n - p\|}{1 + M\|p - Tp\|}\right] \\
&= a(1 - \alpha_n)\|m_n - p\| + a\alpha_n\left[\frac{\varphi(\|0\|) + a\|m_n - p\|}{1 + M\|0\|}\right] \\
&= a(1 - \alpha_n)\|m_n - p\| + a^2\alpha_n\|m_n - p\| \\
&= a(1 - \alpha_n(1 - a))\|m_n - p\| \\
&\precsim a(1 - \alpha(1 - a))\|m_n - p\| \\
&\vdots \\
&\precsim [a(1 - \alpha(1 - a))]^n\|m_1 - p\|.
\end{aligned} \tag{2.23}$$

Let

$$c_n = [a(1 - \alpha(1 - a))]^n\|m_1 - p\|. \tag{2.24}$$

Similarly, using relations (2.22) and (1.3), we have

$$\begin{aligned}
\|x_{n+1}-p\| &= \|Ty_n-p\| \\
&\precsim \frac{\varphi(\|p-Tp\|)+a\|y_n-p\|}{1+M\|p-Tp\|} \\
&= \frac{\varphi(\|0\|)+a\|y_n-p\|}{1+M\|0\|} \\
&= a\|y_n-p\| \\
&= a\|(1-\lambda)x_n+\lambda Tx_n-p\| \\
&\precsim a(1-\lambda)\|x_n-p\|+a\lambda\|Tx_n-p\| \\
&\precsim a(1-\lambda)\|x_n-p\|+a\lambda\left[\frac{\varphi(\|p-Tp\|)+a\|x_n-p\|}{1+M\|p-Tp\|}\right] \\
&= a(1-\lambda)\|x_n-p\|+a\lambda\left[\frac{\varphi(\|0\|)+a\|x_n-p\|}{1+M\|0\|}\right] \\
&= a(1-\lambda)\|x_n-p\|+a^2\lambda\|x_n-p\| \\
&= a(1-\lambda(1-a))\|x_n-p\| \\
&\vdots \\
&\precsim [a(1-\alpha(1-a))]^n\|x_1-p\|.
\end{aligned} \tag{2.25}$$

Let

$$e_n = [a(1-\alpha(1-a))]^n\|x_1-p\|. \tag{2.26}$$

Now, we compute the rate of convergence of the Picard–Mann hybrid iterative process (1.2) and the Picard–Krasnoselskii hybrid iterative process (1.3) as follows:

$$\begin{aligned}
\frac{e_n}{c_n} &= \frac{[a(1-\alpha(1-a))]^n\|x_1-p\|}{[a(1-\alpha(1-a))]^n\|m_1-p\|} \\
&= \frac{\|x_1-p\|}{\|m_1-p\|}.
\end{aligned} \tag{2.27}$$

Clearly, from (2.23) $m_1 \neq p$, so that $0 \prec \|m_1-p\| \prec \infty$. Similarly, from (2.25) $x_1 \neq p$, so that $0 \prec \|x_1-p\| \prec \infty$. Hence,

$$0 < \lim_{n\to\infty}\frac{|\|x_1-p\||}{|\|m_1-p\||} = l < \infty. \tag{2.28}$$

This means that the Picard–Krasnoselskii hybrid iterative process (1.3) has the same rate of convergence as the Picard–Mann hybrid iterative process (1.2). The proof of Proposition 2.2 is completed. □

Remark 2.2. Proposition 2.2 is an extension of the results of Proposition 2.1 to contractive condition satisfying rational expression, which is meaningless in cone metric spaces. This means that our results cannot be deduced in cone metric spaces.

Theorem 2.4. *Let D be a nonempty closed convex subset of a complex valued Banach space $(E, \|.\|)$ and $T : D \to D$ be a contraction mapping satisfying the following contractive* *condition*

$$\|Tx - Ty\| \precsim \frac{\varphi(\|x - Tx\|) + a\|x - y\|}{1 + M\|x - Tx\|}, \quad \forall x, y \in D, \ a \in [0, 1), \quad M \geq 0, \tag{2.29}$$

where $\varphi : \mathbb{C}_+ \to \mathbb{C}_+$ *is a monotone increasing function such that* $\varphi(0) = 0$. *Let* $\{m_n\}$ *be an iterative sequence generated by* (1.2) *with real sequence* $\{\alpha_n\}_{n=0}^{\infty}$ *in* $[0, 1]$ *satisfying* $\sum_{n=0}^{\infty}\alpha_n = \infty$. *Then* $\{m_n\}$ *converges strongly to a unique fixed point of* T.

Proof. We now show that $x_n \to p$ as $n \to \infty$. Using (1.2) and (2.29), we obtain:

$$\begin{aligned}
\|m_{n+1} - p\| &= \|Tz_n - p\| \\
&\precsim \frac{\varphi(\|p - Tp\|) + a\|z_n - p\|}{1 + M\|p - Tp\|} \\
&= \frac{\varphi(\|0\|) + a\|z_n - p\|}{1 + M\|0\|} \\
&= a\|(1 - \alpha_n)m_n + \alpha_n Tm_n - p\| \\
&\precsim a(1 - \alpha_n)\|m_n - p\| + a\alpha_n\|Tm_n - p\| \\
&\precsim a(1 - \alpha_n)\|m_n - p\| + a\alpha_n\left[\frac{\varphi(\|p - Tp\|) + a\|m_n - p\|}{1 + M\|p - Tp\|}\right] \\
&= a(1 - \alpha_n)\|m_n - p\| + a\alpha_n\left[\frac{\varphi(\|0\|) + a\|m_n - p\|}{1 + M\|0\|}\right] \\
&= a(1 - \alpha_n)\|m_n - p\| + a^2\alpha_n\|m_n - p\| \\
&= a(1 - \alpha_n(1 - a))\|m_n - p\|.
\end{aligned} \tag{2.30}$$

Using the fact that $(1 - \alpha_n(1 - a)) < 1$ and $a \in [0, 1)$, we obtain the following inequalities from (2.30).

$$\begin{cases}
\|m_{n+1} - p\| \precsim a(1 - \alpha_n(1 - a))\|m_n - p\| \\
\|m_n - p\| \precsim a(1 - \alpha_{n-1}(1 - a))\|m_{n-1} - p\| \\
\|m_{n-1} - p\| \precsim a(1 - \alpha_{n-2}(1 - a))\|m_{n-2} - p\| \\
\vdots \\
\|m_2 - p\| \precsim a(1 - \alpha_1(1 - a))\|m_1 - p\|.
\end{cases} \tag{2.31}$$

From relation (2.31), we derive

$$\|m_{n+1} - p\| \precsim \|m_1 - p\|a^{n+1}\prod_{k=1}^{n}(1 - \alpha_k(1 - a)), \tag{2.32}$$

where $(1-\alpha_k(1-a)) \in (0, 1)$, since $a \in [0, 1)$ and $\alpha_k \in [0, 1]$ for all $k \in \mathbb{N}$. It is well-known in classical analysis that $1-x \leq e^{-x}$ for all $x \in [0, 1]$. Using these facts together with relation (2.32), we have

$$\|m_{n+1} - p\| \precsim \frac{\|m_1 - p\|a^{n+1}}{e^{(1-a)\sum_{k=1}^{n}\alpha_k}}. \tag{2.33}$$

Therefore,

$$\lim_{n\to\infty} |\|m_{n+1} - p\|| \leq \left\{\frac{|\|m_1 - p\|a^{n+1}|}{|e^{(1-a)\sum_{k=1}^{n}\alpha_k}|}\right\} \longrightarrow 0 \text{ as } n \to \infty. \tag{2.34}$$

This means that $\lim_{n\to\infty}\|m_n - p\| = 0$. That is $m_n \to p$ as $n \to \infty$ as desired.

Next, we show that T has a unique fixed point $p \in F(T) := \{p \in D : Tp = p\}$. Assume that p^* is another fixed point of T, then we have

$$\begin{aligned} \|p - p^*\| &= \|Tp - Tp^*\| \\ &\precsim \frac{\varphi(\|p - Tp\|) + a\|p - p^*\|}{1 + M\|p - Tp\|} \\ &= \frac{\varphi(\|0\|) + a\|p - p^*\|}{1 + M\|0\|} \\ &= a\|p - p^*\|. \end{aligned} \tag{2.35}$$

This implies that $p = p^*$. The proof of Theorem 2.4 is completed. □

Remark 2.3. Theorem 2.4 is an extension of the results of Theorem 2.1 to contractive condition satisfying rational expression, which is meaningless in cone metric spaces. This means that our results cannot be deduced in cone metric spaces.

3. Stability results in complex valued Banach spaces

We begin this section by providing the following numerical example to show that the Picard–Mann hybrid iterative process (1.2) is T-stable.

Example 3.1. Let $E = [0, 1]$. Define $T : [0, 1] \to [0, 1]$ by $Tx = \frac{x}{2}$, where T satisfies contractive condition (1.1), with $\delta = \frac{1}{2}$ and $F(T) = \{0\}$. We now show that the Picard–Mann hybrid iterative scheme (1.2) is T-stable, and hence, almost T-stable. Suppose $\{y_n\} = \frac{1}{n}$ is an arbitrary sequence in E, $p = 0 \in F(T)$ and $\alpha_n = \frac{1}{2}$ for each $n \in \mathbb{N}$.

Then $\lim_{n\to\infty} y_n = 0$. Put

$$\varepsilon_n = |y_{n+1} - f(T, y_n)| = |y_{n+1} - Ta_n|, \tag{3.1}$$

where

$$a_n = (1 - \alpha_n)y_n + \alpha_n T y_n. \tag{3.2}$$

We have,

$$\begin{aligned} \varepsilon_n &= |y_{n+1} - Ta_n| \\ &= |y_{n+1} - \frac{a_n}{2}| \\ &= |y_{n+1} - \frac{(1 - \alpha_n)}{2} y_n - \frac{\alpha_n}{2} . \frac{y_n}{2}| \\ &= |y_{n+1} - \frac{1}{4} y_n - \frac{1}{8} y_n| \\ &= |\frac{1}{n+1} - \frac{1}{4n} - \frac{1}{8n}|. \end{aligned} \tag{3.3}$$

Hence,

$$\lim_{n\to\infty} \varepsilon_n = 0. \tag{3.4}$$

Therefore, the Picard–Mann hybrid iterative process (1.2) is T-stable. Clearly, (1.2) is almost T-stable.

Next, we prove the following stability results for the Picard–Mann hybrid iterative process (1.2).

Theorem 3.1. *Let $(E, \|.\|)$ be a complex valued Banach space and $T : D \subseteq E \to D$ be the contraction mapping defined by (1.1). Suppose there exists $p \in F(T)$ such that the Picard–Mann hybrid iterative process $\{m_n\}_{n=1}^{\infty}$ (1.2) satisfying $\sum_{n=0}^{\infty}\alpha_n = \infty$ and $\alpha_n \leq \alpha \in (0, 1)$ for each $n \in \mathbb{N}$, converges to p. Then*
(1) the Picard–Mann hybrid iterative process (1.2) is T-stable.
(2) the Picard–Mann hybrid iterative process (1.2) is almost T-stable.

Proof. Suppose $\{y_n\}_{n=1}^{\infty} \subset D$ is an arbitrary bounded sequence, put

$$\varepsilon_n = \|y_{n+1} - Ta_n\|, \tag{3.5}$$

where

$$a_n = (1 - \alpha_n)y_n + \alpha_n T y_n. \tag{3.6}$$

Using (1.1), (1.2) and the fact that $\delta \in (0, 1)$, we obtain:

$$\begin{aligned}
\|y_{n+1} - p\| &\precsim \|y_{n+1} - Ta_n\| + \|Ta_n - p\| \\
&\precsim \varepsilon_n + \delta\|a_n - p\| \\
&= \varepsilon_n + \delta\|(1 - \alpha_n)y_n + \alpha_n T y_n - p\| \\
&\precsim \varepsilon_n + \delta(1 - \alpha_n)\|y_n - p\| + \alpha_n\delta\|T y_n - p\| \\
&\precsim \varepsilon_n + \delta(1 - \alpha_n)\|y_n - p\| + \alpha_n\delta^2\|y_n - p\| \\
&\precsim \varepsilon_n + (1 - \alpha_n(1 - \delta))\|y_n - p\|.
\end{aligned} \tag{3.7}$$

Since $\alpha_n \leq \alpha \in (0, 1)$ for all $n \in \mathbb{N}$ and $\delta \in (0, 1)$, we have that $(1 - \alpha_n(1 - \delta)) < 1$. Hence, by Lemma 1.3, relation (3.7) yields:

$$\lim_{n\to\infty} y_n = p. \tag{3.8}$$

Conversely,

$$\begin{aligned}
\varepsilon_n &= \|y_{n+1} - Ta_n\| \\
&\precsim \|y_{n+1} - p\| + \|p - Ta_n\| \\
&\precsim \|y_{n+1} - p\| + \delta\|a_n - p\| \\
&= \|y_{n+1} - p\| + \delta\|(1 - \alpha_n)y_n + \alpha_n T y_n - p\| \\
&\precsim \|y_{n+1} - p\| + \delta(1 - \alpha_n)\|y_n - p\| + \delta\alpha_n\|T y_n - p\| \\
&\precsim \|y_{n+1} - p\| + \delta(1 - \alpha_n)\|y_n - p\| + \delta^2\alpha_n\|y_n - p\| \\
&\precsim \|y_{n+1} - p\| + \delta\|y_n - p\|.
\end{aligned} \tag{3.9}$$

Hence, we have

$$\varepsilon_n \precsim \|y_{n+1} - p\| + \delta\|y_n - p\|. \tag{3.10}$$

Therefore,

$$\lim_{n\to\infty} \varepsilon_n = 0. \tag{3.11}$$

This means that the Picard–Mann hybrid iterative process (1.2) is T-stable.

Next, we prove that iterative process (1.2) is almost T-stable. Suppose that $\sum_{n=1}^{\infty}\varepsilon_n < \infty$, by (3.7) we have

$$\|y_{n+1} - p\| \precsim \varepsilon_n + (1 - \alpha_n(1 - \delta))\|y_n - p\|. \tag{3.12}$$

Hence, by Lemmas 1.1 and 1.4, we have

$$\lim_{n\to\infty} |\|y_n - p\|| = 0. \tag{3.13}$$

This means that $y_n \to p$ as $n \to \infty$.

Conversely, suppose that $\lim_{n\to\infty} y_n = p$, by (3.6) and (3.7), we have

$$\begin{aligned}
\varepsilon_n &\precsim \|y_{n+1} - p\| + \|Ta_n - p\| \\
&\precsim \|y_{n+1} - p\| + \delta\|a_n - p\| \\
&= \|y_{n+1} - p\| + \delta\|(1 - \alpha_n)y_n + \alpha_n Ty_n - p\| \\
&\precsim \|y_{n+1} - p\| + \delta(1 - \alpha_n)\|y_n - p\| + \delta\alpha_n\|Ty_n - p\| \\
&\precsim \|y_{n+1} - p\| + \delta(1 - \alpha_n)\|y_n - p\| + \delta^2\alpha_n\|y_n - p\| \\
&\precsim \|y_{n+1} - p\| + \delta\|y_n - p\|.
\end{aligned} \tag{3.14}$$

Now by Lemmas 1.1 and 1.4, we have that $|\varepsilon_n| \to 0$ as $n \to \infty$. This means that the Picard–Mann hybrid iterative process (1.2) is almost T-stable. The proof of Theorem 3.1 is completed. $\square$

Theorem 3.2. *Let $(E, \|.\|)$ be a complex valued Banach space and $T : D \to D$ be a contraction mapping satisfying the following contractive condition*

$$\|Tx - Ty\| \precsim \frac{\varphi(\|x - Tx\|) + a\|x - y\|}{1 + M\|x - Tx\|}, \quad \forall x, y \in D,\ a \in [0, 1),\quad M \geq 0, \tag{3.15}$$

where $\varphi : \mathbb{C}_+ \to \mathbb{C}_+$ is a monotone increasing function such that $\varphi(0) = 0$. Suppose there exists $p \in F(T)$ such that the Picard–Mann hybrid iterative process $\{m_n\}_{n=1}^{\infty}$ (1.2) satisfying $\sum_{n=0}^{\infty}\alpha_n = \infty$ and $\alpha_n \leq \alpha \in (0, 1)$ for each $n \in \mathbb{N}$, converges to p. Then
(1) the Picard–Mann hybrid iterative process (1.2) is T-stable.
(2) the Picard–Mann hybrid iterative process (1.2) is almost T-stable.

Proof. Suppose $\{g_n\}_{n=1}^{\infty} \subset D$ is an arbitrary bounded sequence, put

$$\varepsilon_n = \|g_{n+1} - Tb_n\|, \tag{3.16}$$

where

$$b_n = (1 - \alpha_n)g_n + \alpha_n Tg_n. \tag{3.17}$$

Using (1.2), (3.15) and the fact that $a \in [0, 1)$, we obtain:

$$\begin{aligned}
\|g_{n+1} - p\| &\precsim \|g_{n+1} - Tb_n\| + \|Tb_n - p\| \\
&\precsim \varepsilon_n + \frac{\varphi(\|p - Tp\|) + a\|b_n - p\|}{1 + M\|p - Tp\|} \\
&= \varepsilon_n + \frac{\varphi(\|0\|) + a\|b_n - p\|}{1 + M\|0\|} \\
&= \varepsilon_n + a\|(1 - \alpha_n)g_n + \alpha_n Tg_n - p\| \\
&\precsim \varepsilon_n + a(1 - \alpha_n)\|g_n - p\| + a\alpha_n\|Tg_n - p\| \\
&\precsim \varepsilon_n + a(1 - \alpha_n)\|g_n - p\| + a\alpha_n\left[\frac{\varphi(\|p - Tp\|) + a\|g_n - p\|}{1 + M\|p - Tp\|}\right] \\
&= \varepsilon_n + a(1 - \alpha_n)\|g_n - p\| + a\alpha_n\left[\frac{\varphi(\|0\|) + a\|g_n - p\|}{1 + M\|0\|}\right] \\
&= \varepsilon_n + a(1 - \alpha_n)\|g_n - p\| + a^2\alpha_n\|g_n - p\| \\
&= \varepsilon_n + a(1 - \alpha_n(1 - a))\|g_n - p\| \\
&\precsim \varepsilon_n + (1 - \alpha_n(1 - a))\|g_n - p\|. \qquad (3.18)
\end{aligned}$$

Since $\alpha_n \le \alpha \in (0, 1)$ for all $n \in \mathbb{N}$ and $a \in [0, 1)$, we have that $(1 - \alpha_n(1 - a)) < 1$. Hence, by Lemma 1.3, relation (3.18) yields:

$$\lim_{n\to\infty} g_n = p. \qquad (3.19)$$

Conversely,

$$\begin{aligned}
\varepsilon_n &= \|g_{n+1} - Tb_n\| \\
&\precsim \|g_{n+1} - p\| + \|p - Tb_n\| \\
&\precsim \|g_{n+1} - p\| + \frac{\varphi(\|p - Tp\|) + a\|b_n - p\|}{1 + M\|p - Tp\|} \\
&= \|g_{n+1} - p\| + \frac{\varphi(\|0\|) + a\|b_n - p\|}{1 + M\|0\|} \\
&= \|g_{n+1} - p\| + a\|(1 - \alpha_n)g_n + \alpha_n Tg_n - p\| \\
&\precsim \|g_{n+1} - p\| + a(1 - \alpha_n)\|g_n - p\| + a\alpha_n\|Tg_n - p\| \\
&\precsim \|g_{n+1} - p\| + a(1 - \alpha_n)\|g_n - p\| + a\alpha_n\left[\frac{\varphi(\|p - Tp\|) + a\|g_n - p\|}{1 + M\|p - Tp\|}\right] \\
&= \|g_{n+1} - p\| + a(1 - \alpha_n)\|g_n - p\| + a^2\alpha_n\|g_n - p\| \\
&= \|g_{n+1} - p\| + a(1 - \alpha_n(1 - a))\|g_n - p\|. \qquad (3.20)
\end{aligned}$$

Since $a(1 - \alpha_n(1 - a)) < 1$, we have

$$\varepsilon_n \precsim \|g_{n+1} - p\| + a(1 - \alpha_n(1 - a))\|g_n - p\|. \qquad (3.21)$$

Therefore,

$$\lim_{n\to\infty} \varepsilon_n = 0. \qquad (3.22)$$

This means that the Picard–Mann hybrid iterative process (1.2) is T-stable.

Next, we prove that iterative process (1.2) is almost T-stable. Suppose that $\sum_{n=1}^{\infty}\varepsilon_n < \infty$, by (3.18) we have

$$\|g_{n+1} - p\| \precsim \varepsilon_n + (1 - \alpha_n(1 - a))\|g_n - p\|. \tag{3.23}$$

Hence, by Lemmas 1.1 and 1.4, we have

$$\lim_{n\to\infty} |\|g_n - p\|| = 0. \tag{3.24}$$

This means that $g_n \to p$ as $n \to \infty$.

Conversely, suppose that $\lim_{n\to\infty} g_n = p$, by (3.17) and (3.18), we have

$$\begin{aligned}
\varepsilon_n &\precsim \|g_{n+1} - p\| + \|Tb_n - p\| \\
&\precsim \|g_{n+1} - p\| + \frac{\varphi(\|p - Tp\|) + a\|b_n - p\|}{1 + M\|p - Tp\|} \\
&= \|g_{n+1} - p\| + \frac{\varphi(\|0\|) + a\|b_n - p\|}{1 + M\|0\|} \\
&= \|g_{n+1} - p\| + a\|(1 - \alpha_n)g_n + \alpha_n T g_n - p\| \\
&\precsim \|g_{n+1} - p\| + a(1 - \alpha_n)\|g_n - p\| + a\alpha_n \left[\frac{\varphi(\|p - Tp\|) + a\|g_n - p\|}{1 + M\|p - Tp\|}\right] \\
&= \|g_{n+1} - p\| + a(1 - \alpha_n)\|g_n - p\| + a\alpha_n \left[\frac{\varphi(\|0\|) + a\|g_n - p\|}{1 + M\|0\|}\right] \\
&= \|g_{n+1} - p\| + a(1 - \alpha_n(1 - a))\|g_n - p\|.
\end{aligned} \tag{3.25}$$

Now by Lemmas 1.1 and 1.4, we have that $|\varepsilon_n| \to 0$ as $n \to \infty$. This means that the Picard–Mann hybrid iterative process (1.2) is almost T-stable. The proof of Theorem 3.2 is completed. □

Remark 3.1. Theorem 3.2 is an extension of the results of Theorem 3.1 to contractive condition satisfying rational expression, which is meaningless in cone metric spaces. This means that our results cannot be deduced in cone metric spaces.

4. Data dependence result

Theorem 4.1. *Let $\tilde{T}$ be an approximate operator of a contraction mapping $T : D \subseteq E \to D$. Let $\{m_n\}_{n=1}^{\infty}$ be an iterative sequence generated by (1.2) for T and define an iterative sequence $\{\tilde{m}_n\}_{n=1}^{\infty}$ as follows*

$$\begin{cases} \tilde{m}_1 = \tilde{m} \in D, \\ \tilde{m}_{n+1} = \tilde{T}\tilde{z}_n, \\ \tilde{z}_n = (1 - \alpha_n)\tilde{m}_n + \alpha_n \tilde{T}\tilde{m}_n, \quad n \in \mathbb{N}, \end{cases} \tag{4.1}$$

with real sequence $\{\alpha_n\}_{n=1}^{\infty}$ in $[0, 1]$ satisfying the following conditions
(i) $\frac{1}{2} \leq \alpha_n$, for all $n \in \mathbb{N}$, and
(ii) $\sum_{n=1}^{\infty}\alpha_n = \infty$.

If $Tp = p$ and $\tilde{T}\tilde{p} = \tilde{p}$ such that $\lim_{n\to\infty}\tilde{m}_n = \tilde{p}$, then we have

$$|\|p - \tilde{p}\|| \leq \frac{5\varepsilon}{1 - \delta},$$

where $\varepsilon > 0$ is a fixed number.

Proof. Using (1.1), (1.2) and (4.1) we have

$$\begin{aligned}\|z_n - \tilde{z}_n\| &\precsim (1-\alpha_n)\|m_n - \tilde{m}_n\| + \alpha_n\|Tm_n - \tilde{T}\tilde{m}_n\| \\ &\precsim (1-\alpha_n)\|m_n - \tilde{m}_n\| + \alpha_n\|Tm_n - T\tilde{m}_n\| + \alpha_n\|T\tilde{m}_n - \tilde{T}\tilde{m}_n\| \\ &\precsim (1-\alpha_n)\|m_n - \tilde{m}_n\| + \alpha_n\delta\|m_n - \tilde{m}_n\| + \alpha_n\varepsilon \\ &= (1-\alpha_n(1-\delta))\|m_n - \tilde{m}_n\| + \alpha_n\varepsilon. \end{aligned} \tag{4.2}$$

$$\begin{aligned}\|m_{n+1} - \tilde{m}_{n+1}\| &= \|Tz_n - T\tilde{z}_n + T\tilde{z}_n - \tilde{T}\tilde{z}_n\| \\ &\precsim \|Tz_n - T\tilde{z}_n\| + \|T\tilde{z}_n - \tilde{T}\tilde{z}_n\| \\ &\precsim \delta\|z_n - \tilde{z}_n\| + \varepsilon. \end{aligned} \tag{4.3}$$

Using (4.2) in (4.3), we have

$$\begin{aligned}\|m_{n+1} - \tilde{m}_{n+1}\| &\precsim \delta[(1-\alpha_n(1-\delta))\|m_n - \tilde{m}_n\| + \alpha_n\varepsilon] + \varepsilon \\ &\precsim \delta(1-\alpha_n(1-\delta))\|m_n - \tilde{m}_n\| + 2\varepsilon. \end{aligned} \tag{4.4}$$

Using the fact that $\delta \in (0,1)$ and $1-\alpha_n \leq \alpha_n$, we have:

$$\begin{aligned}\|m_{n+1} - \tilde{m}_{n+1}\| &\precsim (1-\alpha_n(1-\delta))\|m_n - \tilde{m}_n\| + 2(1-\alpha_n+\alpha_n)\varepsilon \\ &\precsim (1-\alpha_n(1-\delta))\|m_n - \tilde{m}_n\| + (1-\delta)\frac{5\varepsilon}{(1-\delta)}. \end{aligned} \tag{4.5}$$

We denote $\beta_n := \|m_n - \tilde{m}_n\|$, $\mu_n := (1-\delta) \in (0,1)$, $\gamma_n := \frac{5\varepsilon}{(1-\delta)}$.

It follows from Lemma 1.5 that

$$0 \leq \limsup_{n\to\infty} |\|m_n - \tilde{m}_n\|| \leq \limsup_{n\to\infty} \frac{5\varepsilon}{(1-\delta)}. \tag{4.6}$$

From Theorem 2.1, it is known that $\lim_{n\to\infty} m_n = p$. Using this fact together with the assumption that $\lim_{n\to\infty}\tilde{m}_n = \tilde{p}$, we have

$$|\|p - \tilde{p}\|| \leq \frac{5\varepsilon}{(1-\delta)}. \tag{4.7}$$

The proof of Theorem 4.1 is completed. □

5. Applications to delay differential equations

In this section we show that the Picard–Mann hybrid iterative process (1.2) can be used to find the solution of delay differential equations. Let the space $C([a,b])$ with endowed Chebyshev norm

$$\|x - y\|_\infty = \max_{t\in[a,b]} |x(t) - y(t)|e^{ik}, \quad x, y \in C[a,b], \quad k \in [0, \frac{\pi}{2}],$$

denote the space of all continuous complex valued functions on a closed interval $[a,b]$. It is known that $(C([a,b]), \|.\|_\infty)$ is a complex valued Banach space, see Example 1.3.

In this section, we shall study the following delay differential equation.

$$x'(t) = f(t, x(t), x(t-\tau)), \; t \in [t_0, b], \tag{5.1}$$

with initial condition

$$x(t) = \varphi(t), \; t \in [t_0 - \tau, t_0]. \tag{5.2}$$

Assume that the following conditions are satisfied.
(C_1) $t_0, b \in \mathbb{R}$, $\tau > 0$;
(C_2) $f \in C([t_0, b] \times \mathbb{R}^2, \mathbb{R})$;
(C_3) $\varphi \in C([t_0 - \tau, b], \mathbb{R})$;
(C_4) there exist $L_f > 0$ such that

$$|f(t, u_1, u_2) - f(t, v_1, v_2)| \leq L_f \sum_{i=1}^{2} |u_i - v_i|, \quad \forall u_i, v_i \in \mathbb{R}, \\ i = 1, 2, \; t \in [t_0, b]; \tag{5.3}$$

(C_5) $2L_f(b - t_0) < 1$.

By a solution of problem (5.1)–(5.2), we mean a function $x \in C([t_0 - \tau, b], \mathbb{R}) \cap C^1([t_0, b], \mathbb{R})$.

Now, we reformulate problem (5.1)–(5.2) as given in the following integral equation:

$$x(t) = \begin{cases} \varphi(t), & t \in [t_0 - \tau, t_0], \\ \varphi(t_0) + \int_{t_0}^{t} f(s, x(s), x(s - \tau))ds, & t \in [t_0, b]. \end{cases} \tag{5.4}$$

Okeke and Abbas [24] established the following results.

Theorem 5.1. *Assume that conditions (C_1)–(C_5) are satisfied. Then problem (5.1)–(5.2) has a unique solution, say p, in $C([t_0 - \tau, b], \mathbb{R}) \cap C^1([t_0, b], \mathbb{R})$ and the Picard–Krasnoselskii hybrid iterative process (1.3) converges to p.*

Next, we prove the following theorem for the Picard–Mann hybrid iterative process (1.2).

Theorem 5.2. *Assume that conditions (C_1)–(C_5) are satisfied. Then problem (5.1)–(5.2) has a unique solution, say p, in $C([t_0 - \tau, b], \mathbb{R}) \cap C^1([t_0, b], \mathbb{R})$ and the Picard–Mann hybrid iterative process (1.2) converges to p.*

Proof. Let $\{m_n\}_{n=1}^{\infty}$ be the iterative sequence generated by the Picard–Ishikawa hybrid iterative process (1.2) for the operator

$$Tx(t) = \begin{cases} \varphi(t), & t \in [t_0 - \tau, t_0], \\ \varphi(t_0) + \int_{t_0}^{t} f(s, x(s), x(s - \tau))ds, & t \in [t_0, b]. \end{cases} \tag{5.5}$$

Let p denote the fixed point of T. We will prove that $m_n \longrightarrow p$ as $n \to \infty$. It is easy to see that $m_n \to p$ for each $t \in [t_0 - \tau, t_0]$. Now, for each $t \in [t_0, b]$ we have

$$\begin{aligned} \|z_n - p\|_\infty &= \|(1 - \alpha_n)m_n + \alpha_n T m_n - p\|_\infty \\ &\precsim (1 - \alpha_n)\|m_n - p\|_\infty + \alpha_n \|T m_n - T p\|_\infty \\ &\precsim (1 - \alpha_n)\|m_n - p\|_\infty + e^{ik}\alpha_n \max_{t \in [t_0 - \tau, b]} |T m_n(t) - T p(t)| \\ &= (1 - \alpha_n)\|m_n - p\|_\infty + e^{ik}\alpha_n \max_{t \in [t_0 - \tau, b]} |\varphi(t_0) \\ &\quad + \int_{t_0}^{t} f(s, m_n(s), m_n(s - \tau))ds - \end{aligned}$$

$$\begin{aligned}
&\varphi(t_0) - \int_{t_0}^{t} f(s, p(s), p(s-\tau))ds| \\
&= (1-\alpha_n)\|m_n - p\|_\infty + e^{ik}\alpha_n \max_{t\in[t_0-\tau,b]} |\int_{t_0}^{t} f(s, m_n(s), m_n(s-\tau))ds - \\
&\int_{t_0}^{t} f(s, p(s), p(s-\tau))ds| \\
&\precsim (1-\alpha_n)\|m_n - p\|_\infty + e^{ik}\alpha_n \max_{t\in[t_0-\tau,b]} \int_{t_0}^{t} |f(s, m_n(s), m_n(s-\tau)) - \\
&f(s, p(s), p(s-\tau))|ds \\
&\precsim (1-\alpha_n)\|m_n - p\|_\infty + e^{ik}\alpha_n \max_{t\in[t_0-\tau,b]} \int_{t_0}^{t} L_f(|m_n(s) - p(s)| + \\
&|m_n(s-\tau) - p(s-\tau)|)ds \\
&\precsim (1-\alpha_n)\|m_n - p\|_\infty + e^{ik}\alpha_n \int_{t_0}^{t} L_f(\max_{s\in[t_0-\tau,b]}|m_n(s) - p(s)| + \\
&\max_{s\in[t_0-\tau,b]}|m_n(s-\tau) - p(s-\tau)|)ds \\
&\precsim (1-\alpha_n)\|m_n - p\|_\infty + e^{ik}\alpha_n \int_{t_0}^{t} L_f(\|m_n - p\|_\infty + \|m_n - p\|_\infty)ds \\
&\precsim (1-\alpha_n)\|m_n - p\|_\infty + 2\alpha_n L_f(b-t_0)e^{ik}\|m_n - p\|_\infty \\
&\precsim e^{ik}[1 - \alpha_n(1 - 2L_f(b-t_0))]\|m_n - p\|_\infty. \qquad (5.6)
\end{aligned}$$

$$\begin{aligned}
\|m_{n+1} - p\|_\infty &= \|Tz_n - Tp\|_\infty \\
&= e^{ik} \max_{t\in[t_0-\tau,b]} |\int_{t_0}^{t} [f(s, z_n(s), z_n(s-\tau)) - f(s, p(s), p(s-\tau))]ds| \\
&\precsim e^{ik} \max_{t\in[t_0-\tau,b]} \int_{t_0}^{t} |f(s, z_n(s), z_n(s-\tau)) - f(s, p(s), p(s-\tau))|ds \\
&\precsim e^{ik} \max_{t\in[t_0-\tau,b]} \int_{t_0}^{t} L_f(|z_n(s) - p(s)| + |z_n(s-\tau) - p(s-\tau)|)ds \\
&\precsim 2L_f(b-t_0)e^{ik}\|z_n - p\|_\infty. \qquad (5.7)
\end{aligned}$$

Combining (5.6) and (5.7), we have:

$$\|m_{n+1} - p\|_\infty \precsim 2L_f(b-t_0)e^{2ik}[1 - \alpha_n(1 - 2L_f(b-t_0))]\|m_n - p\|. \qquad (5.8)$$

Using assumption (C_5), we have

$$\|m_{n+1} - p\|_\infty \precsim e^{2ik}[1 - \alpha_n(1 - 2L_f(b-t_0))]\|m_n - p\|. \qquad (5.9)$$

Inductively, we obtain

$$\|m_{n+1} - p\|_\infty \precsim e^{2ik}\prod_{j=1}^{n}[1 - \alpha_j(1 - 2L_f(b-t_0))]\|m_1 - p\|. \qquad (5.10)$$

Since $\alpha_n \in [0, 1]$, for all $n \in \mathbb{N}$, using assumption (C_5) yields

$$[1 - \alpha_n(1 - 2L_f(b-t_0))] < 1. \qquad (5.11)$$

Using the same argument as in the proof of Theorem 2.1, we obtain

$$||\|m_{n+1} - p\|_\infty| \leq \frac{|e^{2ik}\|m_1 - p\|_\infty|}{|e^{(1-2L_f(b-t_0))\sum_{n=1}^{\infty}\alpha_n}|}. \tag{5.12}$$

Hence,

$$\lim_{n\to\infty} ||\|m_{n+1} - p\|_\infty| \leq \lim_{n\to\infty} \left\{ \frac{|e^{2ik}\|m_1 - p\|_\infty|}{|e^{(1-2L_f(b-t_0))\sum_{n=1}^{\infty}\alpha_n}|} \right\} \longrightarrow 0 \text{ as } n \to \infty. \tag{5.13}$$

This means that $\lim_{n\to\infty}\|m_n - p\|_\infty = 0$. The proof of Theorem 5.2 is completed. □

Remark 5.1. Theorem 5.2 generalizes and improves several known results in literature including the results of Coman et al. [10] and Okeke and Abbas [24].

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