ARAB JOURNAL OF
MATHEMATICAL SCIENCES

Arab J Math Sci 24 (2) (2018) 147–165

Best proximity pair and fixed point results for noncyclic mappings in modular spaces

KARIM CHAIRA, SAMIH LAZAIZ*

Laboratory of Algebra Analysis and Applications, Department of Mathematics and Computer Sciences, Faculty of Sciences Ben M'Sik, Hassan II University of Casablanca, BP 7955, Sidi Othman, Casablanca, Morocco

Received 17 July 2017; revised 28 January 2018; accepted 8 February 2018
Available online 17 February 2018

Abstract. In this paper, we formulate best proximity pair theorems for noncyclic relatively ρ-nonexpansive mappings in modular spaces in the setting of proximal ρ-admissible sets. As a companion result, we establish a best proximity pair theorem for pointwise noncyclic contractions in modular spaces. To that end, we provide some examples throughout the paper to illustrate the validity of the obtained results.

Keywords: Best proximity pair; Modular spaces; Relatively ρ-nonexpansive mappings; ρ-admissible sets; ρ-normal structure

Mathematics Subject Classification: 47H09; 41A65

1. INTRODUCTION

Let X be an arbitrary vector space.

1. A function $\rho : X \to [0, \infty]$ is called a modular on X if for arbitrary $x, y \in X$,

   (a) $\rho(x) = 0$ if and only if $x = 0$,
   (b) $\rho(\alpha x) = \rho(x)$ for every scalar α with $|\alpha| = 1$,
   (c) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ if $\alpha + \beta = 1$ and $\alpha, \beta \geq 0$.

* Corresponding author.
E-mail addresses: chaira_karim@yahoo.fr (K. Chaira), samih.lazaiz@gmail.com (S. Lazaiz).
Peer review under responsibility of King Saud University.

ELSEVIER Production and hosting by Elsevier

https://doi.org/10.1016/j.ajmsc.2018.02.002

If (c) is replaced by (c)': $\rho(\alpha x+\beta y) \leq \alpha\rho(x)+\beta\rho(y)$ if $\alpha+\beta = 1$ and $\alpha, \beta \geq 0$, we say ρ is convex modular.

2. A modular ρ defines a corresponding modular space, i.e. the vector space X_ρ given by

$$X_\rho = \{x \in X \ : \ \rho(\lambda x) \to 0 \text{ as } \lambda \to 0\}.$$

X_ρ is a linear subspace of X.

The relevance of a best proximity pair, in a couple of non-empty, disjoint subsets A and B of a modular space, is to act as a substitute in the absence of a fixed point. It is also used to provide optimal solutions to the problem of best approximation between two sets.

Eldred, Kirk and Veeramani [7] established the existence of a best proximity pair for noncyclic relatively nonexpansive mappings by using a geometric notion of proximal normal structure in the setting of Banach spaces. The work of the afore-mentioned authors generalizes the notion of normal structure introduced by Milman and Brodskii [6]. Recently, Sankar and Veeramani established the existence and uniqueness of a best proximity pair for noncyclic contraction maps as stated in [18]. Similar results in [1] were discussed by Taghafi and Shahzad who proved the existence of a best proximity pair for a cyclic contraction map in a reflexive Banach space. For other related results, we refer the reader to [1–5,9,10,21,22].

In this paper, we generalize the notion of proximal ρ-normal structure for a ρ-admissible pair (A, B) in modular spaces. We also show that if A and B are proximal ρ-admissible sets, and if the pair (A, B) has proximal ρ-normal structure, then every noncyclic relatively ρ-nonexpansive map has a best proximity pair. As a companion result, we show the existence and uniqueness of a best proximity pair theorem for pointwise noncyclic contractions in the setting of modular spaces.

2. Preliminaries

To describe our results, we need to review some basic definitions and notions related to modular spaces, such as those formulated by Musielak and Orlicz [20]. For further details, we refer the reader to [12,14,16,19]

Definition 1. Let X_ρ be a modular space.

1. We say that (x_n) is ρ-convergent to x and write $x_n \to x\ (\rho)$ if and only if $\rho\,(x_n - x) \to 0$.
2. A sequence (x_n), where $x_n \in X_\rho$, is called ρ-Cauchy if $\rho\,(x_n - x_m) \to 0$ as $n, m \to \infty$.
3. We say that X_ρ is ρ-complete if and only if any ρ-Cauchy sequence in X_ρ is ρ-convergent.
4. A set $C \subset X_\rho$ is called ρ-closed if for any sequence (x_n) of C, the convergence $x_n \to x(\rho)$ implies that x belongs to C.
5. A set $C \subset X_\rho$ is called ρ-sequentially-compact if for any sequence (x_n) of C, there exists a convergent subsequence $\left(x_{n_k}\right)_k$ of (x_n) such that $x_{n_k} \to x(\rho)$ in C.
6. A set $C \subset X_\rho$ is called ρ-bounded if $\sup\{\rho\,(x - y) : x, y \in C\} < \infty$.
7. We will say that ρ satisfies the Fatou property if

$$\rho(x) \leq \liminf_{n\to\infty} \rho(x_n)$$

whenever $x_n \to x\ (\rho)$.

One can check that ρ-balls are ρ-closed if and only if ρ has the Fatou property (cf. [13]).

Definition 2. A pair (A, B) of subsets of X_ρ is said to be a ρ-proximal pair if for each $(x, y) \in A \times B$ there exists $\left(x', y'\right) \in A \times B$ such that

$$\rho\left(x - y'\right) = \rho\left(x' - y\right) = dist_\rho\left(A, B\right).$$

The pair $\left(x, y'\right)$ is said to be proximal in (A, B).

We use (A_0, B_0) to denote the ρ-proximal pair obtained from (A, B) upon setting

$$A_0 = \left\{x \in A : \rho\left(x - y'\right) = dist_\rho\left(A, B\right) \text{ for some } y' \in B\right\}$$

$$B_0 = \left\{y \in B : \rho\left(x' - y\right) = dist_\rho\left(A, B\right) \text{ for some } x' \in A\right\}.$$

A pair (A, B) in a modular space X_ρ is said to satisfy a property if both A and B satisfy that property. For instance, (A, B) is ρ-closed (resp. ρ-bounded) if and only if both A and B are ρ-closed (resp. ρ-bounded); $(A, B) \subset (C, D)$ if and only if $A \subset C$ and $B \subset D$, $(A, B) \neq \emptyset$ if $A \neq \emptyset$ and $B \neq \emptyset$, (A, B) is not reduced to one point means that A and B are not singletons.

Let A, B be nonempty subsets of a modular space X_ρ. We shall adopt the following notations:

$$\delta_\rho\left(A, B\right) = \sup\left\{\rho\left(x - y\right) : x \in A, y \in B\right\}.$$
$$\delta_\rho(x, A) = \sup\left\{\rho\left(x - y\right) : y \in A\right\}, \text{ for all } x \in X_\rho.$$
$$dist_\rho(A, B) = \inf\left\{\rho\left(x - y\right) : x \in A, y \in B\right\}.$$
$$\gamma_\rho(A, B) = \max\left\{\inf\left\{\delta_\rho(x, B) : x \in A\right\}, \inf\left\{\delta_\rho(y, A) : y \in B\right\}\right\}.$$

We introduce some definitions which are in fact extension of the standard definitions in modular space (e.g. see [15, Definition 5.7]). It is worth noting that these notions are more adapted for a pair of subsets (A, B).

Definition 3. Let (A, B) be a ρ-bounded pair.

We will say that (H, K) is a proximal ρ-admissible pair of (A, B) if

$$H = \bigcap_{i \in I} B_\rho\left(y_i, r_i\right) \cap A$$

and

$$K = \bigcap_{i \in I} B_\rho\left(x_i, r_i'\right) \cap B$$

where $(x_i, y_i) \in A \times B$, $r_i, r_i' \geq d_\rho\left(A, B\right)$, I is an arbitrary index set and $B_\rho\left(x, r\right) = \left\{y \in X_\rho : \rho\left(x - y\right) \leq r\right\}$ the standard ρ-closed ball of X_ρ. The family of all proximal ρ-admissible pairs of (A, B) will be denoted by $\mathcal{Q}\left(A, B\right)$.

If $(D_1, D_2) \subseteq (A, B)$, we write

$$co_A^{D_2}\left(D_1\right) = \bigcap_{y \in D_2} B_\rho\left(y, \delta_\rho\left(y, D_1\right)\right) \cap A$$

$$co_B^{D_1}\left(D_2\right) = \bigcap_{x \in D_1} B_\rho\left(x, \delta_\rho\left(x, D_2\right)\right) \cap B.$$

Remark 4. Note that $\left(co_A^{D_2}(D_1), co_B^{D_1}(D_2)\right) \in \mathcal{Q}(A,B)$ and is the smallest ρ-admissible pair of (A,B) which contains (D_1,D_2). Indeed, let $(H,K) \in \mathcal{Q}(A,B)$ such that $(D_1,D_2) \subseteq (H,K)$, then $H = \bigcap_{y\in D_2} B_\rho\left(y, r_y\right) \cap A$, and for each $(x,y) \in D_1 \times D_2$, we have $\rho(x-y) \leq r_y$. Hence, for any $y \in D_2$ we get $\delta_\rho(y, D_1) \leq r_y$ since $D_1 \subseteq H$, which prove that

$$co_A^{D_2}(D_1) = \bigcap_{y\in D_2} B_\rho\left(y, \delta_\rho(y, D_1)\right) \cap A \subseteq \bigcap_{y\in D_2} B_\rho\left(y, r_y\right) \cap A = H.$$

In the same manner, we obtain $co_B^{D_1}(D_2) \subseteq K$.

Definition 5. Let (A,B) be a ρ-bounded pair.

1. $\mathcal{Q}(A,B)$ is said to satisfy the property $(\mathcal{R})$-proximal if for any sequence
$$\left(\{A_n\}_{n\geq 1}, \{B_m\}_{m\geq 1}\right) \subseteq \mathcal{Q}(A,B),$$
which is nonempty and decreasing has a nonempty intersection.
2. $\mathcal{Q}(A,B)$ is said to be proximal ρ-normal, if for each proximal ρ-admissible pair (H,K) not reduced to one point of (A,B) for which $dist_\rho(H,K) = dist_\rho(A,B)$ and $\delta_\rho(H,K) > dist_\rho(H,K)$, there exists $(x,y) \in H \times K$ such that
$$\delta_\rho(x,K) < \delta_\rho(H,K) \text{ and } \delta_\rho(y,H) < \delta_\rho(H,K).$$
3. We say that the pair (A,B) is proximal ρ-sequentially-compact provided that every sequence $\left(\{x_n\}_n, \{y_n\}_n\right)$ of (A,B) satisfying the condition $\rho(x_n - y_n) \to dist_\rho(A,B)$ has a convergent subsequence in (A,B).

Remark 6. Notice that the $\mathcal{Q}(A,A)$ is proximal ρ-normal (resp. has the $(\mathcal{R})$-proximal property) if and only if $\mathcal{Q}(A)$ is ρ-normal (resp. has the $(\mathcal{R})$-property) in the sense of Khamsi and Kozlowski (see [15]).

Definition 7.

1. A map $T: A\cup B \to A\cup B$ will be said
   (a) noncyclic on $A\cup B$ if $T(A) \subseteq A$ and $T(B) \subseteq B$;
   (b) noncyclic relatively ρ-nonexpansive on $A\cup B$ if
      i. T is noncyclic;
      ii. $\rho(Tx - Ty) \leq \rho(x-y)$, for all $(x,y) \in A\times B$.
2. An ordered pair $(a,b) \in A\times B$ is said to be a best proximity pair for the noncyclic mapping T, provided that
$$Ta = a,\ Tb = b \text{ and } \rho(a-b) = dist(A,B).$$

Definition 8. A map $T: A\cup B \to A\cup B$ will be called pointwise noncyclic contraction if

1. T is noncyclic;
2. For each $(x,y) \in A\times B$ there exist $0 \leq \alpha(x), \beta(y) < 1$ such that
$$\rho(Tx - Ty) \leq \alpha(x)\beta(y)\rho(x-y) + (1-\alpha(x))(1-\beta(y))dist_\rho(A,B).$$

Remark 9. Note that every pointwise noncyclic contraction is noncyclic relatively ρ-nonexpansive.

We conclude this section by a modular version of Kirk's fixed point theorem [17] which follows as a corollary of our Theorem 16 (see Corollary 18).

Theorem 10 (*[15, Theorem 5.9]*). *Let A be a ρ-bounded and ρ-closed nonempty subset of X_ρ which satisfies $(\mathcal{R})$-property. Assume that $\mathcal{Q}(A)$ is ρ-normal. If $T : A \to A$ is ρ-nonexpansive, then T has a fixed point.*

3. Noncyclic relatively ρ-nonexpansive mappings

In what follows, we investigate the validity of a technical lemma due to Gillespie and Williams [11], for a pair of ρ-admissible subset in a modular space. This result can be considered the main ingredient of our work and will play an important role in this article.

Lemma 11. *Let (A, B) be a nonempty ρ-bounded pair of X_ρ. Let $T : A \cup B \to A \cup B$ be a noncyclic relatively ρ-nonexpansive mapping. Assume that $\mathcal{Q}(A, B)$ is proximal ρ-normal. Let $(H, K) \in \mathcal{Q}(A, B)$ be a nonempty, not reduced to one point, T-noncyclic pair; i.e., $T(H) \subseteq H$ and $T(K) \subseteq K$ and $dist_\rho(H, K) = dist_\rho(A, B)$. Then, there exists a nonempty T-noncyclic pair $(H_0, K_0) \in \mathcal{Q}(A, B)$ such that $(H_0, K_0) \subseteq (H, K)$ and*

$$\delta_\rho(H_0, K_0) \leq \frac{\delta_\rho(H, K) + \gamma_\rho(H, K)}{2}.$$

Proof. Set $r = \frac{1}{2}\left(\delta_\rho(H, K) + \gamma_\rho(H, K)\right)$. If $\delta_\rho(H, K) = dist_\rho(H, K)$ one can choose $(H_0, K_0) = (H, K)$. We assume that $\delta_\rho(H, K) > dist_\rho(H, K)$. Since $\mathcal{Q}(A, B)$ is proximal ρ-normal, we obtain

$$\gamma_\rho(H, K) < \delta_\rho(H, K)$$

hence $\gamma_\rho(H, K) < r$. Thus, there exists $(x_1, y_1) \in H \times K$ such that

$$\delta(x_1, K) < r \text{ and } \delta(y_1, H) < r.$$

Let

$$D^H = \bigcap_{y \in K} B_\rho(y, r) \cap H$$

$$D^K = \bigcap_{x \in H} B_\rho(x, r) \cap K$$

then $\left(D^H, D^K\right) \neq \emptyset$ since $(x_1, y_1) \in D^H \times D^K$.

Let $\mathcal{F}$ denote the set of all nonempty pairs $\{(E_\alpha, F_\alpha)\}_{\alpha \in \Lambda}$ of $\mathcal{Q}(A, B)$ such that T is noncyclic on $E_\alpha \cup F_\alpha$ and $\left(D^H, D^K\right) \subseteq (E_\alpha, F_\alpha)$ for all $\alpha \in \Lambda$. Obviously, $\mathcal{F}$ is nonempty since $(A, B) \in \mathcal{F}$. Let us define (L_1, L_2) by

$$L_1 = \bigcap_\alpha E_\alpha \text{ and } L_2 = \bigcap_\alpha F_\alpha$$

it is clear that $(L_1, L_2) \neq \emptyset$ since $\left(D^H, D^K\right) \subset (L_1, L_2)$ and T is noncyclic on $L_1 \cup L_2$, thus $(L_1, L_2) \in \mathcal{F}$.

Let $M_1 = D^H \cup T(L_1)$ and $M_2 = D^K \cup T(L_2)$, it is claimed that

$$co_A^{M_2}(M_1) = L_1 \text{ and } co_B^{M_1}(M_2) = L_2.$$

Since $M_1 \subset L_1$, $M_2 \subset L_2$ and the pair (L_1, L_2) is proximal ρ-admissible, then

$$\left(co_A^{M_2}(M_1), co_B^{M_1}(M_2)\right) \subseteq (L_1, L_2),$$

and since $\left(co_A^{M_2}(M_1), co_B^{M_1}(M_2)\right)$ is the smallest ρ-admissible pair which contains (M_1, M_2), as well as

$$T\left(co_A^{M_2}(M_1)\right) \subseteq T(L_1) \text{ and } T\left(co_B^{M_1}(M_2)\right) \subseteq T(L_2)$$

then

$$T\left(co_A^{M_2}(M_1)\right) \subseteq M_1 \text{ and } T\left(co_B^{M_1}(M_2)\right) \subseteq M_2.$$

Note that $dist_\rho(T(L_1), T(L_2)) = dist_\rho(L_1, L_2)$ since T is a relatively ρ-nonexpansive mapping, and since $(M_1, M_2) \subseteq \left(co_A^{M_2}(M_1), co_B^{M_1}(M_2)\right)$ we obtain

$$\left(co_A^{M_2}(M_1), co_B^{M_1}(M_2)\right) \in \mathcal{F}$$

that is

$$\left(co_A^{M_2}(M_1), co_B^{M_1}(M_2)\right) = (L_1, L_2).$$

Set

$$H_0 = \bigcap_{y \in L_2} B_\rho(y, r) \cap L_1$$

$$K_0 = \bigcap_{x \in L_1} B_\rho(x, r) \cap L_2.$$

We claim that (H_0, K_0) is the desired pair. Since $\left(D^H, D^K\right) \subseteq (H_0, K_0)$, then the pair (H_0, K_0) is nonempty. Also $(H_0, K_0) \in \mathcal{Q}(A, B)$.

Note that for each $x \in H_0$ and $y \in K_0$, we have

$$\rho(x - y) \leq r \Rightarrow \delta_\rho(H_0, K_0) \leq r.$$

Next, we show that T is noncyclic on $H_0 \cup K_0$ to complete the proof. Let $y \in K_0$, then

$$\rho(Tx - Ty) \leq \rho(x - y) \leq r \quad (\forall x \in L_1)$$

since T is relatively ρ-nonexpansive. Thus,

$$T(L_1) \subset B_\rho(Ty, r).$$

Recall that $D^H = \bigcap_{y \in K} B_\rho(y, r) \cap H$, then if $z \in D^H$ we have for all $w \in K$

$$\rho(z - w) \leq r$$

and since $(H, K) \in \mathcal{F}$ we get $L_2 \subset K$ then $L_2 \subset B(z, r)$. It is clear that $Ty \in L_2$; that is,

$$Ty \in B_\rho(z, r) \Rightarrow z \in B_\rho(Ty, r)$$

hence, $D^H \subset B_\rho(Ty, r)$, which implies

$$L_1 = co_A^{M_2}\left(D^H \cup T(L_1)\right) \subseteq B_\rho(Ty, r) \cap A.$$

This implies that $Ty \in K_0$; that is, $T(K_0) \subseteq K_0$. Similarly, we can show that $T(H_0) \subseteq H_0$. Since $(L_1, L_2) \subseteq (H, K)$ we get $(H_0, K_0) \subseteq (H, K)$. This completes the proof. □

Definition 12 (*[19]*)**.** Let (A, B) be a pair of nonempty subsets of a modular space X_ρ such that the related A_0 is nonempty. The pair (A, B) is said to have P-property if and only if

$$\begin{cases} \rho(x_1 - y_1) = dist_\rho(A, B) \\ \rho(x_2 - y_2) = dist_\rho(A, B) \end{cases} \Rightarrow \rho(x_1 - x_2) = \rho(y_1 - y_2),$$

where $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$.

Example 13. Let A, B be two nonempty subsets of a modular space X_ρ such that A_0 is nonempty, and $dist_\rho(A, B) = 0$. Then (A, B) has the P-property.

Definition 14 (*[16]*)**.** A modular space X_ρ is said to be strictly convex if for each $x, y \in X_\rho$ such that $\rho(x) = \rho(y)$ and

$$\rho\left(\frac{x+y}{2}\right) = \frac{\rho(x) + \rho(y)}{2}$$

we have $x = y$.

Lemma 15. *Let (A, B) be a nonempty ρ-bounded and convex pair in a strictly convex modular space X_ρ such that ρ is convex. Suppose that A_0 is nonempty, then (A, B) has the P-property.*

Proof. Since A_0 is nonempty, let $y, y' \in B$ and

$$\rho(x - y) = \rho(x' - y') = dist_\rho(A, B)$$

for some $x, x' \in A$. By the convexity of A, B and ρ, we obtain

$$\begin{aligned} dist_\rho(A, B) &\leq \rho\left(\frac{1}{2}(x + x') - \frac{1}{2}(y + y')\right) \\ &\leq \frac{1}{2}\rho(x - y) + \frac{1}{2}\rho(x' - y') = dist_\rho(A, B), \end{aligned}$$

and since X_ρ is strictly convex modular space, we have $x - x' = y - y'$. Hence, (A, B) has the P-property. □

Theorem 16. *Let (A, B) be a nonempty, ρ-bounded and ρ-closed pair in a modular space X_ρ. Assume that A_0 is nonempty. Moreover, assume that $\mathcal{Q}(A, B)$ satisfies the property $(\mathcal{R})$-proximal and has proximal ρ-normal structure. If T is noncyclic relatively ρ-nonexpansive on $A \cup B$ and (A, B) has the P-property, then T has a best proximity pair.*

Proof. Let $\mathcal{F}$ denote the set of all nonempty ρ-closed pairs (E, F) of $\mathcal{Q}(A, B)$ such that T is noncyclic on $E \cup F$ and $dist_\rho(E, F) = d_\rho$ where $d_\rho = dist_\rho(A, B)$. Thus, $\mathcal{F}$ is nonempty since $(A, B) \in \mathcal{F}$.

Define $\tilde{\delta}_\rho : \mathcal{F} \to [0, \infty)$ by

$$\tilde{\delta}_\rho \left(D^A, D^B\right) = \inf\left\{\delta_\rho\left(E, F\right) : (E, F) \in \mathcal{F} \text{ and } (E, F) \subseteq \left(D^A, D^B\right)\right\}.$$

Set $\left(D_1^A, D_1^B\right) = (A, B)$, by definition of $\tilde{\delta}_\rho$, there exists $\left(D_2^A, D_2^B\right) \in \mathcal{F}$ such that $\left(D_2^A, D_2^B\right) \subseteq \left(D_1^A, D_1^B\right)$, $dist_\rho\left(D_2^A, D_2^B\right) = dist_\rho\left(D_1^A, D_1^B\right) = d_\rho$ and

$$\delta_\rho\left(D_2^A, D_2^B\right) < \tilde{\delta}_\rho\left(D_1^A, D_1^B\right) + 1$$

suppose that $\left(D_k^A, D_k^B\right)_{k=1,2,\ldots,n}$ are constructed for $n \geq 1$. Again, by definition of $\tilde{\delta}_\rho$, there exists $\left(D_{n+1}^A, D_{n+1}^B\right) \subseteq \left(D_n^A, D_n^B\right)$ such that

$$\delta_\rho\left(D_{n+1}^A, D_{n+1}^B\right) < \tilde{\delta}_\rho\left(D_n^A, D_n^B\right) + \frac{1}{n}$$

and $dist_\rho\left(D_{n+1}^A, D_{n+1}^B\right) = d_\rho$. Since $\mathcal{Q}(A, B)$ satisfies the property $(\mathcal{R})$-proximal and the sequence

$$\left(\left\{D_n^A\right\}_{n\geq 1}, \left\{D_n^B\right\}_{m\geq 1}\right) \subseteq \mathcal{Q}(A, B),$$

is nonempty and decreasing, then $\left(D_\infty^A, D_\infty^B\right) \neq \emptyset$ where

$$D_\infty^A = \bigcap_{n\geq 1} D_n^A \text{ and } D_\infty^B = \bigcap_{n\geq 1} D_n^B.$$

We also obtain

$$\begin{aligned}
\delta_\rho(A, B) &\leq \delta_\rho\left(D_\infty^A, D_\infty^B\right) \\
&= \inf\{\rho(x-y) : (x, y) \in (\bigcap_{n\geq 1} D_n^A) \times (\bigcap_{m\geq 1} D_m^B)\} \\
&= \inf\{\rho(x-y) : (x, y) \in D_n^A \times D_m^B, \forall (n, m) \in (\mathbb{N}^*)^2\} \\
&= \inf_{n,m\geq 1}\{\rho(x-y) : (x, y) \in D_n^A \times D_m^B\} \\
&\leq \delta_\rho\left(D_n^A, D_n^B\right) \\
&\leq \delta_\rho(A, B).
\end{aligned}$$

Hence, $\delta_\rho\left(D_\infty^A, D_\infty^B\right) = \delta_\rho(A, B)$.

Case 1: If D_∞^A or D_∞^B is reduced to one point, for example $D_\infty^B = \{y\}$ and since $T(D_\infty^B) \subset D_\infty^B$, then $Ty = y$. Since A_0 is nonempty, there exists $x \in A$ such that $\rho(x - y) = dist_\rho(A, B)$. Since T is relatively ρ-nonexpansive on $A \cup B$,

$$\rho(Tx - Ty) = \rho(Tx - y) \leq \rho(x - y) = d_\rho(A, B)$$

by hypothesis, (A, B) has the P-property, then

$$\rho(Tx - y) = \rho(x - y) = d_\rho(A, B) \text{ implies } Tx = x.$$

Similarly, if D_∞^A is reduced to one point.

Case 2: If (D_∞^A, D_∞^B) is not reduced to one point, suppose that

$$\delta_\rho\left(D_\infty^A, D_\infty^B\right) = dist_\rho\left(D_\infty^A, D_\infty^B\right)$$

For each $\left(x, x', y\right) \in (D_\infty^A)^2 \times D_\infty^B$, we get

$$\rho(x - y) = \rho\left(x' - y\right) = dist_\rho\left(D_\infty^A, D_\infty^B\right).$$

Since (A, B) has the P-property, we have $\rho(x-x') = \rho(y-y) = 0$. Then, D_{∞}^{A} is a singleton. Similarly, we can show that D_{∞}^{B} is a singleton. This implies that the noncyclic mapping T has a best proximity pair in this case. Now, assume that

$$\delta_\rho\left(D_{\infty}^{A}, D_{\infty}^{B}\right) > dist_\rho\left(D_{\infty}^{A}, D_{\infty}^{B}\right).$$

Using Lemma 11, there exists $\left(D_A^*, D_B^*\right) \subseteq \left(D_{\infty}^{A}, D_{\infty}^{B}\right)$

$$\delta_\rho\left(D_A^*, D_B^*\right) \leq \frac{\delta_\rho\left(D_{\infty}^{A}, D_{\infty}^{B}\right) + \gamma_\rho\left(D_{\infty}^{A}, D_{\infty}^{B}\right)}{2} \tag{1}$$

which implies

$$\begin{aligned}
\delta_\rho\left(D_A^*, D_B^*\right) &\leq \delta_\rho\left(D_{\infty}^{A}, D_{\infty}^{B}\right)\\
&\leq \delta_\rho\left(D_n^{A}, D_n^{B}\right)\\
&\leq \tilde{\delta}_\rho\left(D_n^{A}, D_n^{B}\right) + \frac{1}{n}\\
&\leq \delta_\rho\left(D_A^*, D_B^*\right) + \frac{1}{n} \quad \text{since } \left(D_A^*, D_B^*\right) \subseteq \left(D_n^{A}, D_n^{B}\right)
\end{aligned}$$

for any $n \geq 1$. If we let $n \to \infty$, we get $\delta_\rho\left(D_A^*, D_B^*\right) = \delta_\rho\left(D_{\infty}^{A}, D_{\infty}^{B}\right)$. By (1) we get

$$\delta_\rho\left(D_{\infty}^{A}, D_{\infty}^{B}\right) \leq \gamma_\rho\left(D_{\infty}^{A}, D_{\infty}^{B}\right)$$

this contradicts the assumption that $\mathcal{Q}(A, B)$ is proximal ρ-normal. This completes the proof. □

Example 17. Let the real space $X = \{x = (x_n)_{n\geq 1} \in \mathbb{R}^{\mathbb{N}^*} : \sum_{n\geq 1}|x_n|^{\frac{1}{2}} < \infty\}$, and define the modular functional $\rho : X \to [0, \infty]$ by

$$\rho(x) = \sum_{n=1}^{\infty} |x_n|^{\frac{1}{2}}, \quad \text{for all } x = (x_n)_{n\geq 1} \in X.$$

Suppose that $\{e_n\}$ is the canonical basis of X and let

$$A = \{e_3 + \frac{1}{2}e_1\} \cup \{e_3 + e_n \ : \ n \in \mathbb{N} \setminus \{0, 1, 3\}\} \ \text{ and } \ B = \{e_1, e_3\}.$$

Then, (A, B) is ρ-bounded, ρ-closed in X_ρ and not convex. A is not ρ-sequentially-compact because the sequence $\{e_3 + e_n\}_{n\neq 3}$ does not have any ρ-convergent subsequence.

Let $u = e_3 + \frac{1}{2}e_1$ in A, we have $\rho(u - e_3) = \sqrt{\frac{1}{2}}$. Also, for all $x \in A$, $\rho(x - e_1) \geq \sqrt{\frac{1}{2}}$ and $\rho(x - e_3) \geq \sqrt{\frac{1}{2}}$, which implies that $dist_\rho(A, B) = \sqrt{\frac{1}{2}}$.

$\mathcal{Q}(A, B)$ satisfies the property $(\mathcal{R})$-proximal, indeed, let $\left(\{H_n\}_{n\geq 1}, \{K_m\}_{m\geq 1}\right)$ be a sequence of $\mathcal{Q}(A, B)$ which is nonempty and decreasing.

1. If for each $n \in \mathbb{N}^*$, $H_n = \bigcap_{i\in I_n} B_\rho(e_1, r_{i,n}) \cap A$, we get for all $i \in I_n$, $r_{i,n} \geq 1 + \sqrt{\frac{1}{2}}$, because $H_n \neq \emptyset$ for any $n \in \mathbb{N}^*$ and, since

$$\rho(e_3 + \frac{1}{2}e_1 - e_1) = 1 + \sqrt{\frac{1}{2}}$$

we obtain $e_3 + \frac{1}{2}e_1 \in B_\rho(e_1, 1 + \sqrt{\frac{1}{2}}) \cap A \subset \bigcap_{n\geq 1} H_n$. Hence,

$$\bigcap_{n\geq 1} H_n \neq \emptyset.$$

2. If for each $n \in \mathbb{N}^*$, $H_n = \bigcap_{j\in J_n} B_\rho(e_3, r'_{j,n}) \cap A$, where $r'_{j,n} \geq dist_\rho(A, B)$, for all $j \in J_n$, so $e_3 + \frac{1}{2}e_1 \in B_\rho(e_3, \sqrt{\frac{1}{2}}) \cap A \subset \bigcap_{n\geq 1} H_n$, because $\rho(e_3 + \frac{1}{2}e_n - e_3) = \sqrt{\frac{1}{2}}$. Hence, $\bigcap_{n\geq 1} H_n \neq \emptyset$.
3. If there exists $n \in \mathbb{N}^*$ such that

$$H_n = \left(\bigcap_{i\in I_n} B_\rho(e_1, r_{i,n})\right) \cap \left(\bigcap_{j\in J_n} B_\rho(e_3, r'_{j,n})\right) \cap A,$$

we have $e_3 + \frac{1}{2}e_1 \in B_\rho(e_1, 1+\sqrt{\frac{1}{2}}) \cap B_\rho(e_3, \sqrt{\frac{1}{2}}) \cap A \subset \bigcap_{n\geq 1} H_n$. Hence, $\bigcap_{n\geq 1} H_n \neq \emptyset$.

Since, for each $n \in \mathbb{N}^*$, K_n is equal to $\{e_1\}$ or $\{e_3\}$ or B, so $\bigcap_{n\geq 1} K_n \neq \emptyset$.

$\mathcal{Q}(A, B)$ has the proximal ρ-normal structure. Indeed, let (H, K) be a proximal ρ-admissible pair of (A, B) not reduced to one point for which $dist_\rho (H, K) = dist_\rho (A, B) = \sqrt{\frac{1}{2}}$ and $\delta_\rho (H, K) > dist_\rho (H, K)$. So, $K = B$ and $e_3 + \frac{1}{2}e_1 \in H$. Therefore, $\delta_\rho \left(e_3 + \frac{1}{2}e_1, K\right) = 1+\sqrt{\frac{1}{2}}$. Since H is not reduced to one point, there exists $m \in \mathbb{N}\setminus\{0, 1, 3\}$ such that $e_3 + e_m \in H$ and $\delta_\rho (H, K) \geq \rho(e_3 + e_m - e_1) = 3$. Hence, $\delta_\rho (H, K) > \max\{\delta_\rho \left(e_3 + \frac{1}{2}e_1, K\right), \delta_\rho (e_3, H)\}$.

Let $T : A \cup B \to A \cup B$ be a mapping defined by

$$Ty = e_3 \text{ if } y \in B \text{ and } Tx = \begin{cases} u \text{ if } x = u \\ v \text{ if } x \in A \setminus \{u\}, \text{ where } v = e_3 + e_2. \end{cases}$$

T is noncyclic and

$$\rho(Tu - Ty) = \rho(u - e_3) = \sqrt{\frac{1}{2}} \leq \rho(u - y), \text{ for each } y \in B,$$

$$\rho(Tx - Ty) = \rho(v - e_3) = 1 \leq \rho(x - y), \text{ for each } x \in A \setminus \{u\} \text{ and } y \in B.$$

Then, T is noncyclic relatively ρ-nonexpansive on $A \cup B$. Therefore, all assumptions of Theorem 16 are satisfied, so T has a best proximity pair; namely,

$$Tu = u, \; Te_3 = e_3 \text{ and } \rho(u - e_3) = dist_\rho(A, B).$$

Corollary 18. *Let A be a ρ-bounded and ρ-closed nonempty subset of X_ρ. Assume that $\mathcal{Q}(A, A)$ is ρ-normal and satisfies the property $(\mathcal{R})$-proximal. If $T : A \to A$ is ρ-nonexpansive, then T has a fixed point.*

The following lemma will be useful.

Lemma 19. *Let (A, B) be a nonempty ρ-bounded and proximal ρ-sequentially-compact pair in a modular space X_ρ for which ρ satisfies the Fatou property. Then (A_0, B_0) is nonempty, ρ-sequentially-compact and $dist_\rho (A_0, B_0) = dist_\rho (A, B)$.*

Proof. It is obvious that

$$dist_\rho (A_0, B_0) = dist_\rho (A, B).$$

Let (x_n) and (y_n) be two sequences in A and B respectively, such that

$$\rho (x_n - y_n) \to dist_\rho (A, B).$$

Since (A, B) is a proximal ρ-compact pair, there exist subsequences $\left(x_{n_k}\right)$ and $\left(y_{n_k}\right)$ of (x_n) and (y_n) respectively, such that $x_{n_k} \to x \in A$ and $y_{n_k} \to y \in B$ as $k \to \infty$. Since ρ is Fatou, then

$$\rho(x - y) \leq \liminf_k \rho\left(x_{n_k} - y_{n_k}\right) = dist_\rho(A, B).$$

This implies that A_0 is nonempty since $x \in A_0$. Similarly, we can see that B_0 is nonempty. The ρ-sequential-compactness of A_0 is vacuous since each sequence (x_n) of A_0 has a convergent subsequence for which this limit is in A_0 because A_0 is ρ-closed in A. Indeed, let $(x_n) \subset A_0$ such that $x_{n_k} \to a$, then there exists a sequence (y_n) in B_0 such that

$$\rho(x_n - y_n) \to dist_\rho(A, B).$$

The proximal ρ-compactness of (A, B) implies the existence of subsequences $\left(x_{n_k}\right)$ and $\left(y_{n_k}\right)$ of (x_n) and (y_n), respectively, such that $x_{n_k} \to x \in A$ and $y_{n_k} \to y \in B$. Since ρ is Fatou, so,

$$\rho(x - y) \leq \liminf_k \rho\left(x_{n_k} - y_{n_k}\right) = dist_\rho(A, B)$$

then $x \in A_0$, the uniqueness of the limit implies that $x = a$. Hence (A_0, B_0) is a ρ-sequentially-compact pair. □

If we replace the assumption X_ρ has $(\mathcal{R})$-proximal property and A_0 is nonempty by the condition (A, B) is a proximal ρ-sequentially-compact pair in Theorem 16, we obtain the following result.

Theorem 20. *Let (A, B) be a nonempty, ρ-bounded, ρ-closed and proximal ρ-sequentially-compact pair in a modular space X_ρ for which ρ satisfies the Fatou property. Moreover, assume that $\mathcal{Q}(A, B)$ has the proximal ρ-normal structure. If T is noncyclic relatively ρ-nonexpansive on $A \cup B$ and (A, B) has the P-property, then T has a best proximity pair.*

Proof. Let $\mathcal{F}$ denote the set of all nonempty ρ-closed pairs (E, F) of $\mathcal{Q}(A, B)$ such that T is noncyclic on $E \cup F$ and $\rho(x - y) = d_\rho$ for some $(x, y) \in E \times F$ where $d_\rho = dist_\rho(A, B)$. Thus, $\mathcal{F}$ is nonempty since $(A, B) \in \mathcal{F}$.

Define $\tilde{\delta}_\rho : \mathcal{F} \to [0, \infty)$ by

$$\tilde{\delta}_\rho\left(D^A, D^B\right) = \inf\left\{\delta_\rho(E, F) : (E, F) \in \mathcal{F} \text{ and } (E, F) \subseteq \left(D^A, D^B\right)\right\}.$$

Set $\left(D_1^A, D_1^B\right) = (A, B)$, by definition of $\tilde{\delta}_\rho$, there exists $\left(D_2^A, D_2^B\right) \in \mathcal{F}$ such that $\left(D_2^A, D_2^B\right) \subseteq \left(D_1^A, D_1^B\right)$, $dist_\rho\left(D_2^A, D_2^B\right) = dist_\rho\left(D_1^A, D_1^B\right) = d_\rho$ and

$$\delta_\rho\left(D_2^A, D_2^B\right) < \tilde{\delta}_\rho\left(D_1^A, D_1^B\right) + 1$$

suppose that $\left(D_k^A, D_k^B\right)_{k=1,2,\ldots,n}$ are constructed for $n \geq 1$. Again, by definition of $\tilde{\delta}_\rho$, there exists $\left(D_{n+1}^A, D_{n+1}^B\right) \subseteq \left(D_n^A, D_n^B\right)$ such that

$$\delta_\rho\left(D_{n+1}^A, D_{n+1}^B\right) < \tilde{\delta}_\rho\left(D_n^A, D_n^B\right) + \frac{1}{n}$$

and $dist_\rho\left(D_{n+1}^A, D_{n+1}^B\right) = d_\rho$. Using Lemma 19, (A_0, B_0) is ρ-sequentially-compact, then $\left(D_\infty^A, D_\infty^B\right) \neq \emptyset$ where

$$D_\infty^A = \bigcap_{n \geq 1} D_n^A \text{ and } D_\infty^B = \bigcap_{n \geq 1} D_n^B.$$

Indeed, one can choose two sequences (x_n) and (y_n) such that $(x_n, y_n) \in D_n^A \times D_n^B$ for each $n \geq 1$ and

$$\rho(x_n - y_n) = d_\rho$$

using the same method as before, there exists $\left(x_{n_k}\right)$ of (x_n) and $\left(y_{n_k}\right)$ of (y_n) such that $x_{n_k} \to x\,(\rho)$ and $y_{n_k} \to y\,(\rho)$. Let $p \geq 1$ and define two subsets of A_0 and B_0 as follows

$$C_p^A = \left\{x_{n_k} : k \geq p\right\} \text{ and } C_p^B = \left\{y_{n_k} : k \geq p\right\}$$

hence $x \in \bigcap_p C_p^A$ and $y \in \bigcap_p C_p^B$. Thus, $x \in \bigcap_{n\geq 1} D_n^A = \bigcap_{k\geq 1} D_{n_k}^A$ and $y \in \bigcap_{n\geq 1} D_n^B = \bigcap_{k\geq 1} D_{n_k}^B$. Also, since ρ satisfies the Fatou property we get

$$\rho(x - y) = d_\rho\left(D_\infty^A, D_\infty^B\right) = d_\rho(A, B).$$

Note that, $T\left(D_\infty^A\right) = T\left(\bigcap_n D_n^A\right) \subseteq \bigcap_n T\left(D_n^A\right) \subseteq \bigcap_n D_n^A = D_\infty^A$, in the same manner $T\left(D_\infty^B\right) \subseteq D_\infty^A$ and, $\left(D_\infty^A, D_\infty^B\right) \in \mathcal{Q}(A, B)$ since $\left(D_n^A, D_n^B\right) \in \mathcal{Q}(A, B)$ for all $n \geq 1$, then $\left(D_\infty^A, D_\infty^B\right) \in \mathcal{F}$.

Case 1: If D_∞^A or D_∞^B is reduced to one point, for example $D_\infty^B = \{y\}$ and since $T(D_\infty^B) \subset D_\infty^B$, we have $Ty = y$. Also, $\rho(x - y) = d_\rho\left(D_\infty^A, D_\infty^B\right) = d_\rho(A, B)$, for some $x \in D_\infty^A$. Since T is relatively ρ-nonexpansive on $D_\infty^A \cup D_\infty^B$,

$$\rho(Tx - Ty) = \rho(Tx - y) \leq \rho(x - y) = d_\rho(A, B)$$

by hypothesis, (A, B) has the P-property, then

$$\rho(Tx - y) = \rho(x - y) = d_\rho(A, B) \text{ implies } Tx = x.$$

Similarly, If D_∞^A is reduced to one point.

Case 2: If (D_∞^A, D_∞^B) is not reduced to one point.

In this step, we can use the same argument as in Theorem 16 to prove that

$$\delta_\rho\left(D_\infty^A, D_\infty^B\right) = dist_\rho\left(D_\infty^A, D_\infty^B\right),$$

hence we get for each $(x, y) \in D_\infty^A \times D_\infty^B$,

$$Tx = x,\ Ty = y \text{ and } \rho(x - y) = dist_\rho\left(D_\infty^A, D_\infty^B\right),$$

which completes the proof. □

Example 21. Let $X = \mathbb{R}$ and define the modular functional $\rho : X \to [0, \infty[$ by

$$\rho(x) = |x|^{\frac{1}{3}}, \quad \text{for all } x \in \mathbb{R}.$$

Define

$$A = \{-\pi\} \cup [-\frac{\pi}{2}, 0] \text{ and } B = [2, 3] \cup \{4\}$$

(A, B) is a nonempty, ρ-bounded, ρ-closed and proximal ρ-sequentially-compact pair in a modular space X_ρ. (A, B) is not a convex pair and $dist_\rho(A, B) = 2^{\frac{1}{3}}$. Note that $\mathcal{Q}(A, B)$ has the proximal ρ-normal structure and the P-property.

Define $T : A \cup B \to A \cup B$ by:

$$\begin{cases} Tx = \dfrac{x + \sin(x)}{2} & \text{if } x \in A \\ Ty = \dfrac{y + 2}{2} & \text{if } y \in B. \end{cases}$$

We have

$$\begin{aligned}\rho(Tx - Ty) &= \left|\frac{1}{2}(x-y) + \frac{1}{2}(\sin(x) - 2)\right|^{\frac{1}{3}} \\ &= \left(\frac{1}{2}(y-x) + \frac{1}{2}(2 - \sin(x))\right)^{\frac{1}{3}} \\ &\leq \rho(x-y).\end{aligned}$$

So, T is noncyclic relatively ρ-nonexpansive on $A \cup B$ and has a best proximity pair:

$$T0 = 0,\ T2 = 2 \text{ and } \rho(0-2) = dist_\rho(A, B).$$

The following example shows that the proximal ρ-normal structure of Theorem 20 is a necessary assumption to get the existence of a best proximity pair of noncyclic relatively ρ-nonexpansive maps.

Example 22. Let $X = \mathbb{R}^2$ and define the modular functional $\rho : X \to [0, \infty[$ by

$$\rho(x) = |x_1|^{\frac{1}{3}} + |x_2|^{\frac{1}{3}}, \quad \text{for all } x = (x_1, x_2) \in \mathbb{R}^2.$$

Define

$$A = \{(1, 0), (1, 1)\} \text{ and } B = \{(2, 0), (2, 1)\}$$

(A, B) is a nonempty, ρ-bounded, ρ-closed and ρ-sequentially-compact pair (so proximal ρ-sequentially-compact pair) in a modular space X_ρ. A and B are not convex sets. We have $dist_\rho(A, B) = 1$ and (A, B) has the P-property.

Define $T : A \cup B \to A \cup B$ by:

$$\begin{cases} T(1,0) = (1,1) \\ T(1,1) = (1,0) \end{cases} \quad \text{and} \quad \begin{cases} T(2,0) = (2,1) \\ T(2,1) = (2,0). \end{cases}$$

We have

$$\rho(Tx - Ty) \leq \rho(x - y), \text{ for all } (x, y) \in A \times B,$$

that is, T is noncyclic relatively ρ-nonexpansive on $A \cup B$. However, T has no best proximity pair. Note that $\mathcal{Q}(A, B)$ is not proximal ρ-normal, since for $(H, K) = (A, B)$,

$$dist_\rho(H, K) = dist_\rho(A, B) \text{ and } \delta_\rho(H, K) = 2 > dist_\rho(H, K) = 1,$$

but

$$\delta_\rho((1, 0), K) = 2 = \delta_\rho(H, K) \text{ and } \delta_\rho((1, 1), K) = 2 = \delta_\rho(H, K).$$

If we set $A = B$, we get the ρ-sequentially compact version of Theorem 10.

Corollary 23. *Let A be a ρ-bounded and ρ-sequentially compact nonempty subset of X_ρ satisfying the Fatou property. Assume that $\mathcal{Q}(A, A)$ is ρ-normal. If $T : A \to A$ is ρ-nonexpansive, then T has a fixed point.*

Corollary 24. *Let (A, B) be as Theorem 20 and let $T : A \cup B \to A \cup B$ be a pointwise noncyclic contraction, then T has a best proximity pair.*

Remark 25. If we do not use the technical Lemma 11, Zorn's Lemma will guarantee the existence of diametral pairs for noncyclic relatively ρ-nonexpansive mappings. Recall that an ordered pair (x^*, y^*) belonging to $L_1 \times L_2$ with $\rho(x^* - y^*) = dist_\rho(L_1, L_2)$ is called a diametral pair if

$$\delta_\rho(x^*, L_2) = \delta_\rho(y^*, L_1) = \delta_\rho(L_1, L_2).$$

For more details see [8, Lemma 4.3]

Theorem 26. *Let (A, B) be a nonempty, ρ-bounded and ρ-closed pair in a modular space X_ρ. Assume $\mathcal{Q}(A, B)$ is compact and A_0 is nonempty. Assume ρ satisfies the Fatou property. Let T be a noncyclic relatively ρ-nonexpansive on $A \cup B$. Then, there exists a nonempty ρ-closed pair (L_1, L_2) of $\mathcal{Q}(A, B)$, which is T-noncyclic and satisfies $dist_\rho(L_1, L_2) = dist_\rho(A, B)$. Moreover, each $(x^*, y^*) \in L_1 \times L_2$ with $\rho(x^* - y^*) = dist_\rho(A, B)$ is a diametral pair.*

Proof. Let $\mathcal{F}$ denote the collection of all nonempty and ρ-closed pairs (E, F) of $\mathcal{Q}(A, B)$ such that T is noncyclic on $E \cup F$ and $dist_\rho(E, F) = dist_\rho(A, B)$. $\mathcal{F}$ is nonempty since $(A, B) \in \mathcal{F}$.

Also, $\mathcal{F}$ is partially ordered by reverse inclusion, let $\{(E_\alpha, F_\alpha)\}_{\alpha \in \Lambda}$ be a descending chain in $\mathcal{F}$ and define (E, F) by

$$E = \bigcap_\alpha E_\alpha \text{ and } F = \bigcap_\alpha F_\alpha$$

$(E, F) \neq \emptyset$, since $\mathcal{Q}(A, B)$ is compact and T is noncyclic on $E \cup F$, and $dist_\rho(E, F) = dist_\rho(A, B)$.

So, every increasing chain in $\mathcal{F}$ is bounded above with respect to reverse inclusion relation. Then, using Zorn's Lemma there exists a minimal element for $\mathcal{F}$, say (L_1, L_2).

Assume that there exists a pair $(x^*, y^*) \in L_1 \times L_2$ with $\rho(x^* - y^*) = dist_\rho(A, B)$ which is not a diametral pair. Then

$$\min\{\delta_\rho(x^*, L_2), \delta_\rho(y^*, L_1)\} < \delta_\rho(L_1, L_2).$$

Set $r_1 = \delta_\rho(x^*, L_2) \leq \delta_\rho(L_1, L_2)$ and $r_2 = \delta_\rho(y^*, L_1) < \delta_\rho(L_1, L_2)$. and let

$$D^{L_1} = \bigcap_{y \in L_2} B_\rho(y, r_1) \cap L_1$$

and

$$D^{L_2} = \bigcap_{x \in L_1} B_\rho(x, r_2) \cap L_2$$

then $dist_\rho\left(D^{L_1}, D^{L_2}\right) = dist_\rho(A, B)$ and $\left(D^{L_1}, D^{L_2}\right) \neq \emptyset$ since $(x^*, y^*) \in D^{L_1} \times D^{L_2}$.

Let $M_1 = T(L_1)$ and $M_2 = T(L_2)$, it is claimed that

$$co_{L_1}^{M_2}(M_1) = L_1 \text{ and } co_{L_2}^{M_1}(M_2) = L_2.$$

Indeed, we have

$$\left(co_{L_1}^{M_2}(M_1), co_{L_2}^{M_1}(M_2)\right) \subseteq (L_1, L_2)$$

then

$$T\left(co_{L_1}^{M_2}(M_1)\right) \subseteq M_1 \text{ and } T\left(co_{L_2}^{M_1}(M_2)\right) \subseteq M_2$$

and since $(M_1, M_2) \subseteq \left(co_{L_1}^{M_2}(M_1), co_{L_2}^{M_1}(M_2)\right)$ we get

$$T\left(co_{L_1}^{M_2}(M_1)\right) \subseteq co_{L_1}^{M_2}(M_1) \text{ and } T\left(co_{L_2}^{M_1}(M_2)\right) \subseteq co_{L_2}^{M_1}(M_2).$$

Since $T(L_1) \times T(L_2) \subset co_{L_1}^{M_2}(M_1) \times co_{L_2}^{M_1}(M_2)$ and $dist_\rho(L_1, L_2) = dist_\rho(A, B)$, we get

$$dist_\rho\left(co_{L_1}^{M_2}(M_1), co_{L_2}^{M_1}(M_2)\right) = dist_\rho(A, B).$$

Thus,

$$\left(co_{L_1}^{M_2}(M_1), co_{L_2}^{M_1}(M_2)\right) \in \mathcal{F}$$

that is

$$\left(co_{L_1}^{M_2}(M_1), co_{L_2}^{M_1}(M_2)\right) = (L_1, L_2).$$

We have for each $(x, y) \in D^{L_1} \times D^{L_2}$,

$$L_1 = co_{L_1}^{M_2}(M_1) \subset B_\rho(y, r_2) \text{ and } L_2 = co_{L_2}^{M_1}(M_2) \subset B_\rho(x, r_1). \tag{2}$$

Moreover, T is noncyclic on $D^{L_1} \cup D^{L_2}$. Indeed, let $w \in D^{L_2}$, for each $x \in L_1$ we have $\rho(w - x) \leq r_2$. Since T is relatively ρ-nonexpansive,

$$\rho(Tw - Tx) \leq \rho(w - x) \leq r_2, \ \forall x \in L_1.$$

Thus,

$$T(L_1) \subset B_\rho(Tw, r_2).$$

Note that $co_{L_1}^{M_2}(M_1) \subseteq \bigcap_{y \in L_2} B_\rho\left(Ty, \delta_\rho(Ty, T(L_1))\right)$. If $x \in L_1$ and since $w \in L_2$,

$$\rho(x - Tw) \leq \delta_\rho(Tw, T(L_1)) \leq \delta_\rho(w, L_1),$$

because T is relatively ρ-nonexpansive. So

$$\forall x \in L_1, \ \rho(x - Tw) \leq r_2$$

hence, $Tw \in D^{L_2}$. Then $T(D^{L_2}) \subset D^{L_2}$. Similarly, $T(D^{L_1}) \subset D^{L_1}$. That is T is noncyclic on $D^{L_1} \cup D^{L_2}$.

Since $(x^*, y^*) \in D^{L_1} \times D^{L_2}$ and $\rho(x^* - y^*) = dist_\rho(A, B)$, we get

$$dist_\rho\left(D^{L_1}, D^{L_2}\right) = dist_\rho(A, B)$$

it follows that $\left(D^{L_1}, D^{L_2}\right) \in \mathcal{F}$, the minimality of (L_1, L_2) implies that $L_1 = D^{L_1}$ and $L_2 = D^{L_2}$. Thereby,

$$\delta_\rho(L_1, L_2) = \delta_\rho(L_1, D^{L_2}) = \sup\{\delta_\rho(y, L_1) \ : \ y \in D^{L_2}\} \leq r_2$$

which is contradiction. This completes the proof. □

4. Pointwise noncyclic contraction

In this section, we give a best proximity pair result for pointwise noncyclic contraction in the setting of modular spaces. Note that the proof is done directly and without the notion of proximal ρ-normal structure.

Theorem 27. *Let (A, B) be a nonempty, ρ-bounded and ρ-closed pair in a modular space X_ρ. Assume $\mathcal{Q}(A, B)$ is compact and ρ satisfies the Fatou property. If $T : A \cup B \to A \cup B$ is a pointwise noncyclic contraction and (A, B) has the P-property, then T has a unique best proximity pair.*

Proof. Using Zorn's Lemma and compactness of $\mathcal{Q}(A, B)$, we obtain a nonempty, ρ-bounded and ρ-closed pair (L_1, L_2) in X_ρ which is minimal with respect to being invariant under the noncyclic mapping T and $dist_\rho(L_1, L_2) = dist_\rho(A, B)$. So, we must have $co_{L_1}^{M_2}(M_1) = L_1$ and $co_{L_2}^{M_1}(M_2) = L_2$. Let $(x, y) \in L_1 \times L_2$, there exist $0 \le \alpha(x), \beta(x) < 1$ such that

$$\rho(Tx - Ty) \le \alpha(x)\beta(y)\rho(x - y) + (1 - \alpha(x))(1 - \beta(y))dist_\rho(A, B).$$

We have,

$$\rho(Tx - Ty) \le \alpha(x)\delta_\rho(x, L_2) + (1 - \alpha(x))dist_\rho(A, B)$$

$$\rho(Tx - Ty) \le \beta(y)\delta_\rho(y, L_1) + (1 - \beta(y))dist_\rho(A, B),$$

and so,

$$T(L_2) \subset B_\rho\left(Tx, \alpha(x)\delta_\rho(x, L_2) + (1 - \alpha(x))dist_\rho(A, B)\right)$$

$$T(L_1) \subset B_\rho\left(Ty, \beta(y)\delta_\rho(y, L_1) + (1 - \beta(y))dist_\rho(A, B)\right).$$

Therefore,

$$L_2 = co_{L_2}^{M_1}(M_2) \subset B_\rho\left(Tx, \alpha(x)\delta_\rho(x, L_2) + (1 - \alpha(x))dist_\rho(A, B)\right)$$

$$L_1 = co_{L_1}^{M_2}(M_1) \subset B_\rho\left(Ty, \beta(y)\delta_\rho(y, L_1) + (1 - \beta(y))dist_\rho(A, B)\right),$$

where $M_1 = T(L_1)$ and $M_2 = T(L_2)$. Hence,

$$\delta_\rho(Tx, L_2) \le \alpha(x)\delta_\rho(x, L_2) + (1 - \alpha(x))dist_\rho(A, B) \tag{3}$$

$$\delta_\rho(Ty, L_1) \le \beta(y)\delta_\rho(y, L_1) + (1 - \beta(y))dist_\rho(A, B). \tag{4}$$

Now, let $(x^*, y^*) \in L_1 \times L_2$ be a fixed element. Put

$$r_1 = \alpha(x^*)\delta_\rho(x^*, L_2) + (1 - \alpha(x^*))dist_\rho(A, B)$$

$$r_2 = \beta(y^*)\delta_\rho(y^*, L_1) + (1 - \beta(y^*))dist_\rho(A, B).$$

and let $dist_\rho(A, B) \le r_1 \le r_2$. Set

$$D^{L_1} = \bigcap_{y \in L_2} B_\rho(y, r_2) \cap L_1$$

$$D^{L_2} = \bigcap_{x \in L_1} B_\rho(x, r_1) \cap L_2.$$

It follows from (3) that $\delta_\rho(Tx^*, L_2) \le r_1 \le r_2$ and by using (4) we have $\delta_\rho(Ty^*, L_1) \le r_2$, that is $(Tx^*, Ty^*) \in D^{L_1} \times D^{L_2}$. Also, if $x \in D^{L_1}$, then $\delta(x, L_2) \le r_2$. It follows

$$\delta(Tx, L_2) \le \alpha(x)\delta_\rho(x, L_2) + (1 - \alpha(x))dist_\rho(A, B) \le \delta(x, L_2) \le r_2$$

$$\delta(Ty, L_1) \le \beta(y)\delta_\rho(y, L_1) + (1 - \beta(y))dist_\rho(A, B) \le \delta(y, L_1) \le r_1,$$

which implies $Tx \in D^{L_1}$ and $Ty \in D^{L_1}$, so $T(D^{L_1}) \subset D^{L_1}$ and $T(D^{L_2}) \subset D^{L_2}$. Thus, T is noncyclic on $D^{L_1} \cup D^{L_2}$, and since (D^{L_1}, D^{L_2}) is a ρ-bounded and ρ-closed pair in X_ρ, from the minimality of (L_1, L_2) we get $L_1 = D^{L_1}$ and $L_2 = D^{L_2}$. Thereby, for all $x \in L_1$,

$$\begin{aligned}\delta_\rho(x, L_2) &\le \alpha(x^*)\delta_\rho(x^*, L_2) + (1 - \alpha(x^*))dist_\rho(A, B)\\ &\le \alpha(x^*)\delta_\rho(L_1, L_2) + (1 - \alpha(x^*))dist_\rho(A, B).\end{aligned}$$

This leads to

$$\begin{aligned}\delta_\rho(L_1, L_2) &= \sup_{x \in L_1} \delta_\rho(x, L_2)\\ &\le \alpha(x^*)\delta_\rho(L_1, L_2) + (1 - \alpha(x^*))dist_\rho(A, B).\end{aligned}$$

Hence,

$$\delta_\rho(L_1, L_2) = dist_\rho(A, B).$$

Since (A, B) has the P-property, we conclude that (L_1, L_2) are singletons and so T has a best proximity pair, say $(p, q) \in L_1 \times L_2$. If $(p', q') \in A \times B$ is another best proximity pair, then

$$\begin{aligned}\rho(p - q') &= \rho(Tp - Tq')\\ &\le \alpha(p)\beta(q')\rho(p - q') + (1 - \alpha(p))(1 - \beta(q'))dist_\rho(A, B),\end{aligned}$$

which implies that $\rho(p - q') = dist_\rho(A, B)$ and since (A, B) has the P-property, we have $q = q'$. Similarly, $p = p'$, which completes the proof. □

We conclude this paper by the following example which shows how the P-property is a necessary condition to ensure the existence of a best proximity pair for pointwise noncyclic contractions in Theorem 27.

Example 28. Let the real space $X = \{x = (x_n)_{n \ge 1} \in \mathbb{R}^{\mathbb{N}^*} : \sum_{n \ge 1} |x_n|^{\frac{1}{2}} < \infty\}$, and define the modular functional $\rho : X \to [0, \infty]$ by

$$\rho(x) = \max\{r(x), 2\|x\|_\infty\} \text{ for all } x = (x_n)_{n \ge 1} \in X$$

where, $\|.\|_\infty$ denotes the ℓ_∞-norm and $r : x \mapsto \sum_{n=1}^{\infty} |x_n|^{\frac{1}{2}}$ the modular functional of X. Suppose that $\{e_n\}$ is the canonical basis of X. Define

$$A = \{x = (x_n)_{n \ge 1} : x_3 = 1, \rho(x) \le 2\} \text{ and } B = \{y_1 = e_1 + e_2, y_2 = e_1 - e_2\}.$$

Then, (A, B) is ρ-bounded, ρ-closed in X_ρ and B is not convex. A is not ρ-sequentially-compact because the sequence $\{e_3 + e_n\}_{n \ne 3}$ does not have any ρ-convergent subsequence.

Notice that $u = e_1 + e_3$ and $v = e_2 + e_3$ are two points of A, so $\rho(u - v) = \rho(v - y_1) = 2$. Moreover, for each $x = (x_1, x_2, 1, x_4, \ldots) \in A$ we have $r(x) \leq 2$ which implies that $\sum_{n \neq 3} |x_n|^{\frac{1}{2}} \leq 1$, so $|x_n| \leq 1$, for all $n \geq 1$. Thus, for all $x \in A$, $\rho(x - y_1) \geq 2$ and $\rho(x - y_2) \geq 2$ which implies that $dist_\rho(A, B) = 2$.

$\mathcal{Q}(A, B)$ is compact. Indeed, let $\left(\{H_\alpha\}_{\alpha \in \Lambda}, \left\{K_\beta\right\}_{\beta \in \Gamma}\right)$ be a family of $\mathcal{Q}(A, B)$ such that $\left(\cap_{\alpha \in \Lambda_1} H_\alpha, \cap_{\beta \in \Gamma_1} K_\alpha\right) \neq \emptyset$, for any finite subsets $\Lambda_1 \subset \Lambda$ and $\Gamma_1 \subset \Gamma$.

1. If for each $\alpha \in \Lambda$, $H_\alpha = \bigcap_{i \in I_\alpha} B_\rho(y_1, r_{i,\alpha}) \cap A$, where $r_{i,\alpha} \geq dist_\rho(A, B)$, for all $i \in I_\alpha$, so $B_\rho(y_1, 2) \cap A \subset \bigcap_{\alpha \in \Lambda} H_\alpha$ and since $e_3 + e_1 \in B_\rho(y_1, 2) \cap A$, we have $\bigcap_{\alpha \in \Lambda} H_\alpha \neq \emptyset$.
2. If for each $\alpha \in \Lambda$, $H_\alpha = \bigcap_{j \in J_\alpha} B_\rho(y_2, r_{j,\alpha}) \cap A$, where $r_{j,\alpha} \geq dist_\rho(A, B)$, for all $j \in J_\alpha$, so $B_\rho(y_2, 2) \cap A \subset \bigcap_{\alpha \in \Lambda} H_\alpha$ and since $e_3 + e_1 \in B_\rho(y_2, 2) \cap A$, we have $\bigcap_{\alpha \in \Lambda} H_\alpha \neq \emptyset$.
3. If there exists $\alpha \in \Lambda$ such that $H_\alpha = \left(\bigcap_{i \in I_\alpha} B_\rho(y_1, r_{i,\alpha})\right) \cap \left(\bigcap_{j \in J_\alpha} B_\rho(y_2, r'_{j,\alpha})\right) \cap A$. We have $e_3 + e_1 \in B_\rho(y_1, r_{i,\alpha}) \cap B_\rho(y_2, r'_{j,\alpha}) \cap A \subset \bigcap_{\alpha \in \Lambda} H_\alpha$, hence $\bigcap_{\alpha \in \Lambda} H_\alpha \neq \emptyset$.

Since, for each $\beta \in \Gamma$, K_β is equal to $\{y_1\}$ or $\{y_2\}$ or B we have $\bigcap_{\beta \in \Gamma} K_\beta \neq \emptyset$.

Let $T : A \cup B \to A \cup B$ be a mapping defined by

$$T y_i = y_1, \text{ for } i \in \{1, 2\} \text{ and } Tx = \begin{cases} v \text{ if } x = u \\ u \text{ if } x \in A \setminus \{u\} \end{cases}$$

Then, T is noncyclic and for each $k \in [0, 1)$, $x \in A$ and $i \in \{1, 2\}$, we have

$$\rho(Tx - Ty_i) = 2 = 2k + 2(1 - k) \leq k\rho(x - y_i) + (1 - k)dist_\rho(A, B),$$

therefore, T is a pointwise noncyclic contraction. Nevertheless, T has no best proximity pair since (A, B) does not satisfy the P-property, $\rho(u - y_1) = \rho(v - y_1) = 2$ but

$$\rho(u - v) \neq 0.$$

Acknowledgments

The authors would like to thank all the members of the editorial board who were responsible for dealing with this paper and the anonymous referee for their valuable comments and suggestions improving the content of this paper.

Competing interests

The authors declare that they have no competing interests.

References

[1] M.A. Al-Thagafi, N. Shahzad, Best proximity sets and equilibrium pairs for a finite family of multimaps, Fixed Point Theory Appl. 2008 (1) (2008) 1–10.
[2] M.A. Al-Thagafi, N. Shahzad, Best proximity pairs for Kakutani multimaps, Nonlinear Anal. 70 (2009) 1209–1216.
[3] M.A. Al-Thagafi, N. Shahzad, Convergence and existence results for best proximity points, Nonlinear Anal. 70 (2009) 3665–3671.
[4] S.S. Basha, N. Shahzad, R. Jeyaraj, Best proximity point theorems for reckoning optimal approximate solutions, Fixed Point Theory Appl. 2012 (2012) 202.
[5] S.S. Basha, N. Shahzad, R. Jeyaraj, Best proximity points : approximation and optimisation, Optim. Lett. 7 (2013) 145–155.

[6] M. Derafshpour, S. Rezapour, N. Shahzad, Best proximity points of cyclic ϕ-contractions in ordered metric spaces, Topol. Methods Nonlinear Anal. 37 (2013) 193–202.

[7] A.A. Eldred, W.A. Kirk, P. Veeramani, Proximal normal structure and relatively non-expansive mappings, Studia Math. 171 (2005) 283–293.

[8] M. Gabeleh, Semi-normal structure and best proximity pair results in convex metric spaces, Banach J. Math. Anal. 8 (2) (2014) 214–228.

[9] M. Gabeleh, N. Shahzad, Best proximity pair and fixed point results for noncyclic mappings in convex metric spaces, Filomat 30 (12) (2016) 3149–3158.

[10] M. Gabeleh, N. Shahzad, Seminormal structure and fixed point of cyclic relatively nonexpansive mappings, Abstr. Appl. Anal. 2014 (2014) 8 pages.

[11] A. Gillespie, B. Williams, Fixed point theorem for nonexpansive mappings on Banach spaces, Appl. Anal. 9 (1979) 121–124.

[12] M.A. Khamsi, A convexity property in modular function spaces, Math. Japon. 44 (2) (1990) 269–280.

[13] M.A. Khamsi, Uniform noncompact convexity, fixed point property in modular spaces, Math. Japon. 41 (1) (1994) 1–6.

[14] M.A. Khamsi, Fixed point theory in modular function spaces, in: Proceeding of the Workshop on Recent Advances on Metric Fixed Point Theory held in Sevilla, September, 1995, pp. 31–55.

[15] M.A. Khamsi, W.M. Kozlowski, Fixed Point Theory in Modular Functions Spaces, Birkhauser, 2015.

[16] M.A. Khamsi, W.M. Kozlowski, S. Reich, Fixed point property in modular function spaces, Nonlinear Anal. 14 (11) (1990) 935–953.

[17] W.A. Kirk, Nonexpansive mappings in metric and Banach spaces, Milan J. Math. 51 (1) (1981) 133–144.

[18] G.S.R. Kosuru, P. Veeramani, A note on existence and convergence of best proximity points for pointwise cyclic contractions, Numer. Funct. Anal. Optim. 82 (2011) 821–830.

[19] J. Musielak, Orlicz Spaces and Modular Spaces, in: Lecture notes in Math., vol. 1034, Springer-Verlag, Berlin, Heidelberg, New York, 1983.

[20] J. Musielak, W. Orlicz, On modular space, Studia Math. 18 (1959) 591–597.

[21] H.K. Pathak, N. Shahzad, Some results on best proximity point for cyclic mappings, Bull. Belg. Math. Soc. Simon Stevin 20 (2013) 559–572.

[22] Sh. Rezapour, M. Derafshpour, N. Shahzad, Best proximity point of cyclic ϕ-contractions on reflexive Banach, Fixed Point Theory Appl. 2010 (1) (2010) 946178.