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An extension of the reflexive property of rings

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Abstract. Mason introduced the notion of reflexive property of rings as a generalization of reduced rings. For a ring endomorphism α, Krempa studied α-rigid rings as an extension of reduced rings. In this note, we introduce the notion of α-quasi reflexive rings as a generalization of α-rigid rings and a natural extension of the reflexive property to ring endomorphisms. We investigate various properties of these rings and also study ring theoretic extensions such as polynomial rings, trivial extensions, right (left) quotient rings, Dorroh extensions etc. over these rings.

Keywords: α-quasi reflexive ring; Matrix ring; Polynomial ring; Right (left) quotient ring; Dorroh extension

Mathematics Subject Classification: 16W20; 16S99

1. INTRODUCTION

Throughout this article, all rings are associative with unity unless otherwise explicitly mentioned and all ring endomorphisms are nonzero. Given a ring R, the polynomial ring with an indeterminate x over R is denoted by $R[x]$, the n by n full (resp., upper triangular) matrix ring over R by $M_n(R)$ (resp., $U_n(R)$), center of R by $Z(R)$, E_{ij} denotes the matrix with (i, j)-entry 1 and other entries 0, and $\mathbb{Z}_n$ denotes the ring of integers modulo n.

A subgroup H of a group G is normal if and only if for $a, b \in G$, $ab \in H$ implies $ba \in H$. For arbitrary subsets of semigroups and rings, Thierrin [16] called this property as *réflectif*.

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Motivated by this, Mason in 1981 introduced the notion of reflexive property for ideals. Due to Mason [14], a right ideal I of a ring R is called *reflexive* if for $a, b \in R$, $aRb \subseteq I$ implies $bRa \subseteq I$ and a ring R is called *reflexive* if the zero ideal of R is reflexive. From the definition, it is clear that every commutative ring is reflexive. Also semiprime rings are reflexive by an easy computation [14].

Following the literature, a ring R is called *reduced* if it has no nonzero nilpotent elements. A ring R is called *reversible* [3] if for $a, b \in R$, $ab = 0$ implies $ba = 0$. Due to Bell [2], a ring R is called an *IFP* (*insertion-of-factors-property*) ring if for $a, b \in R$, $ab = 0$ implies $aRb = 0$. A ring R is called *abelian* if each idempotent of R is central. The relations among the classes of rings mentioned above are as follows.

$$\begin{array}{ccccccc} \text{Reduced} & \Rightarrow & \text{Reversible} & \Rightarrow & \text{IFP} & \Rightarrow & \text{Abelian} \\ & & \Downarrow & & & & \\ & & \text{Reflexive} & & & & \end{array}$$

Krempa [10] extended the notion of reduced rings to ring endomorphisms. Due to Krempa [10], an endomorphism α of a ring R is called *rigid* if for $a \in R$, $a\alpha(a) = 0$ implies $a = 0$ and a ring R is called *α-rigid* [7] if there exists a rigid endomorphism α of R. Following [7, pp. 218], any rigid endomorphism is injective and α-rigid rings are reduced. Also from [1, Lemma 2.1(iii)], a ring R is α-rigid if and only if for $a \in R$, $\alpha(a)a = 0$ implies $a = 0$. Following [12], an endomorphism α of a ring R is called *right* (resp., *left*) *skew reflexive* if for $a, b \in R$, $aRb = 0$ implies $bR\alpha(a) = 0$ (resp., $\alpha(b)Ra = 0$) and a ring R is called *right* (resp., *left*) *α-skew reflexive* if there exists a right (resp., left) skew reflexive endomorphism α of R. A ring R is called *α-skew reflexive* [12] if it is both right and left α-skew reflexive. Note that α-rigid rings are right α-skew reflexive by [12, Proposition 2.6].

Motivated by above, for a ring endomorphism α, we introduce the notion α-quasi reflexive rings as a generalization of α-rigid rings and a natural extension of the reflexive property to ring endomorphisms. We begin with the following definition.

Definition 1.1. (1) An endomorphism α of a ring R is called *right* (resp., *left*) *quasi reflexive* if for $a, b \in R$, $aR\alpha(b) = 0$ implies $bR\alpha(a) = 0$ (resp., $\alpha(a)Rb = 0$ implies $\alpha(b)Ra = 0$).

(2) A ring R is called *right* (resp., *left*) *α-quasi reflexive* if there exists a right (resp., left) quasi reflexive endomorphism α of R.

(3) A ring R is called *α-quasi reflexive* if it is both right and left α-quasi reflexive.

Remark 1.2. (1) A ring R is reflexive if it is right (left) 1_R-quasi reflexive where 1_R denotes the identity endomorphism of R.

(2) Any domain R is α-quasi reflexive for any monomorphism α of R.

We provide non-trivial examples of α-quasi reflexive rings as follows.

Proposition 1.3. *Let R be a reduced ring. Then*

$$S = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} : a, b \in R \right\}$$

is α-quasi reflexive where $\alpha : S \to S$ is defined by

$$\alpha\left(\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\right) = \begin{pmatrix} a & -b \\ 0 & a \end{pmatrix}.$$

Proof. Assume that $A, B \in S$ such that $AS\alpha(B) = 0$ where

$$A = \begin{pmatrix} a_1 & b_1 \\ 0 & a_1 \end{pmatrix} \text{ and } B = \begin{pmatrix} a_2 & b_2 \\ 0 & a_2 \end{pmatrix}.$$

Then $A\alpha(B) = 0$ and so we have

$$a_1 a_2 = 0. \tag{1}$$

$$-a_1 b_2 + b_1 a_2 = 0. \tag{2}$$

From (1), we have $a_2 a_1 = 0$ since R is reduced (and so reversible). Multiplying (2) by a_2 from left and using $a_2 a_1 = 0$, we have $a_2 b_1 a_2 = 0$. This gives $(a_2 b_1)^2 = 0$ and so $a_2 b_1 = 0$ as R is reduced. Thus $b_1 a_2 = 0$ and $b_2 a_1 = 0$. Then for any $\left(\begin{smallmatrix} r & s \\ 0 & r \end{smallmatrix}\right) \in S$, we have

$$\begin{pmatrix} a_2 & b_2 \\ 0 & a_2 \end{pmatrix}\begin{pmatrix} r & s \\ 0 & r \end{pmatrix}\begin{pmatrix} a_1 & -b_1 \\ 0 & a_1 \end{pmatrix} = \begin{pmatrix} a_2 r a_1 & -a_2 r b_1 + a_2 s a_1 + b_2 r a_1 \\ 0 & a_2 r a_1 \end{pmatrix} = 0$$

as R being a reduced ring, is IFP. Thus $BS\alpha(A) = 0$ and so S is right α-quasi reflexive. In a similar way, we can show that S is left α-quasi reflexive and hence S is α-quasi reflexive. □

The condition "R is a reduced ring" in Proposition 1.3 cannot be replaced by "R is a reversible ring" by the following example.

Example 1.4. We refer to the ring in [9, Example 2.1]. Let $A = \mathbb{Z}_2\{a_0, a_1, a_2, b_0, b_1, b_2, c\}$ be the free algebra with zero constant terms in noncommuting indeterminates $a_0, a_1, a_2, b_0, b_1, b_2, c$ over $\mathbb{Z}_2$. Note that A is a ring without unity and consider an ideal of the ring $\mathbb{Z}_2 + A$, say I, generated by

$$\begin{aligned} &a_0 b_0,\ a_0 b_1 + a_1 b_0,\ a_0 b_2 + a_1 b_1 + a_2 b_0,\ a_1 b_2 + a_2 b_1,\ a_2 b_2,\ a_0 r b_0,\ a_2 r b_2, \\ &b_0 a_0,\ b_0 a_1 + b_1 a_0,\ b_0 a_2 + b_1 a_1 + b_2 a_0,\ b_1 a_2 + b_2 a_1,\ b_2 a_2,\ b_0 r a_0,\ b_2 r a_2, \\ &(a_0 + a_1 + a_2) r (b_0 + b_1 + b_2),\ (b_0 + b_1 + b_2) r (a_0 + a_1 + a_2), \text{ and } r_1 r_2 r_3 r_4, \end{aligned}$$

where $r, r_1, r_2, r_3, r_4 \in A$. Then clearly $A^4 \subseteq I$. Let $R = (\mathbb{Z}_2 + A)/I$. By [9, Example 2.1], R is reversible. For simplicity, we identify the elements of $\mathbb{Z}_2 + A$ with their images in R. Let

$$S = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} : a, b \in R \right\}$$

and $\alpha : S \to S$ be defined by

$$\alpha\left(\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\right) = \begin{pmatrix} a & -b \\ 0 & a \end{pmatrix}.$$

Let

$$A = \begin{pmatrix} a_0 & a_1 \\ 0 & a_0 \end{pmatrix} \text{ and } B = \begin{pmatrix} b_0 c & -b_1 c \\ 0 & b_0 c \end{pmatrix}.$$

Clearly, $A, B \in S$ and for any $\left(\begin{smallmatrix} r & s \\ 0 & r \end{smallmatrix}\right) \in S$, we have

$$\begin{pmatrix} a_0 & a_1 \\ 0 & a_0 \end{pmatrix}\begin{pmatrix} r & s \\ 0 & r \end{pmatrix}\begin{pmatrix} b_0 c & b_1 c \\ 0 & b_0 c \end{pmatrix} = \begin{pmatrix} 0 & (a_0 r b_1 + a_1 r b_0)\, c \\ 0 & 0 \end{pmatrix}. \tag{3}$$

For $r = \gamma + h$, where $\gamma \in \mathbb{Z}_2$ and $h \in A$, we have

$$(a_0 r b_1 + a_1 r b_0)\, c = \gamma\, (a_0 b_1 + a_1 b_0)\, c + (a_0 h b_1 + a_1 h b_0)\, c = 0$$

by the construction of I. Therefore from (3), we obtain $AS\alpha(B) = 0$. However,

$$B\alpha(A) = \begin{pmatrix} b_0c & -b_1c \\ 0 & b_0c \end{pmatrix} \begin{pmatrix} a_0 & -a_1 \\ 0 & a_0 \end{pmatrix} = \begin{pmatrix} 0 & -(b_0ca_1 + b_1ca_0) \\ 0 & 0 \end{pmatrix} \neq 0$$

as $b_0ca_1 + b_1ca_0 \neq 0$, entailing $BS\alpha(A) \neq 0$. Therefore S is not right α-quasi reflexive and so S is not α-quasi reflexive.

Proposition 1.5. *For a ring R with an endomorphism α, if R is α-rigid, then R is α-quasi reflexive.*

Proof. Let R be α-rigid and let $a, b \in R$ such that $aR\alpha(b) = 0$. Then $a\alpha(b) = 0$ and so we have $bRa\alpha(bRa) = bR(a\alpha(b))\alpha(Ra) = 0$. Since R is α-rigid, therefore $bRa = 0$. This gives $ba = 0$ and so $ab = 0$ as α-rigid rings are reduced by [7, pp. 218]. Thus

$$bR\alpha(a)\alpha(bR\alpha(a)) = bR\alpha(ab)\alpha(R\alpha(a)) = 0,$$

entailing $bR\alpha(a) = 0$ as R is α-rigid. Therefore R is right α-quasi reflexive. Using the fact that a ring R is α-rigid if and only if for $a \in R$, $\alpha(a)a = 0$ implies $a = 0$ (by [1, Lemma 2.1(iii)]), we can show that R is left α-quasi reflexive by similar arguments. Hence R is α-quasi reflexive. □

Of course by the help of Proposition 1.3, one can easily conclude that converse of Proposition 1.5 need not be true. This shows that the notion of α-quasi reflexive rings is a proper generalization of that of α-rigid rings. Next we show that the notions of α-quasi reflexive and α-skew reflexive rings are independent of each other.

Example 1.6. (1) Consider a ring $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ with usual addition and multiplication. Let $\alpha : R \to R$ be an endomorphism defined by $\alpha((a, b)) = (b, a)$. For $(a, b), (c, d) \in R$, $(a, b)R\alpha((c, d)) = 0$ implies $a\mathbb{Z}_2d = 0 = b\mathbb{Z}_2c$, yielding $(c, d)R\alpha((a, b)) = 0$. Therefore R is right α-quasi reflexive. In a similar way, we can show that R is left α-quasi reflexive and so R is α-quasi reflexive. However, R is neither right nor left α-skew reflexive by means of [12, Example 2.7(1)].

(2) Let F be a field. Then $\alpha : F[x] \to F[x]$ defined by $\alpha(f(x)) = f(0)$ is an endomorphism. Since $F[x]$ is a domain, therefore it is α-skew reflexive by [12, pp. 219]. For $f(x) = 1 + x$, $g(x) = x \in F[x]$, $f(x)F[x]\alpha(g(x)) = 0$ but $g(x)\alpha(f(x)) = x \neq 0$, entailing $g(x)F[x]\alpha(f(x)) \neq 0$. Therefore $F[x]$ is not right α-quasi reflexive. Similarly, we can show that $F[x]$ is not left α-quasi reflexive.

Proposition 1.7. *Let R be a reflexive ring and α an endomorphism of R. Then R is right α-quasi reflexive if and only if R is left α-quasi reflexive.*

Proof. Let R be right α-quasi reflexive and let $a, b \in R$ such that $\alpha(a)Rb = 0$. Since R is reflexive, therefore $bR\alpha(a) = 0$. Again R is right α-quasi reflexive and so $aR\alpha(b) = 0$. Again using the reflexive property of R, we obtain $\alpha(b)Ra = 0$. Therefore R is left α-quasi reflexive. Converse can be proved similarly. □

The condition "R is a reflexive ring" in Proposition 1.7 is not superfluous by the following example.

Example 1.8. The argument here is due to [12, Example 2.2]. Let S be a reflexive ring. Consider a ring $R = U_2(S)$. By [11, Example 2.7], R is not reflexive.

(1) Let $\alpha : R \to R$ be an endomorphism defined by

$$\alpha\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}.$$

Assume that $AR\alpha(B) = 0$ for

$$A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \text{ and } B = \begin{pmatrix} a' & b' \\ 0 & c' \end{pmatrix} \in R.$$

Then for any $\left(\begin{smallmatrix} r & s \\ 0 & t \end{smallmatrix}\right) \in R$, we have $ara' = 0$ and so $aSa' = 0$. Since S is reflexive, therefore $a'Sa = 0$ and so $BR\alpha(A) = 0$. Thus R is right α-quasi reflexive.

For

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \in R,$$

we have $\alpha(A)RB = 0$ but

$$\alpha(B)A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \neq 0,$$

entailing $\alpha(B)RA \neq 0$. Therefore R is not left α-quasi reflexive.

(2) Let $\alpha' : R \to R$ be an endomorphism defined by

$$\alpha'\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix}.$$

By applying arguments similar to (1), we can show that R is left α'-quasi reflexive which is not right α'-quasi reflexive.

From Example 1.8, it is clear that a right (left) α-quasi reflexive ring need not be abelian. Also from Example 1.6(2) and 1.8, we can conclude that the notions of reflexive and right (left) α-quasi reflexive rings are independent of each other. However, we have the following.

Proposition 1.9. *For a ring R with an endomorphism α such that $\alpha^2 = 1_R$, the following are equivalent:*

(1) *R is reflexive.*

(2) *R is right α-quasi reflexive.*

(3) *R is left α-quasi reflexive.*

Proof. Since α is an endomorphism of R such that $\alpha^2 = 1_R$ so it is clear that α is bijective.

(1) $\Rightarrow$ (2) Let R be reflexive and let $a, b \in R$ such that $aR\alpha(b) = 0$. Then for any $r \in R$, there exists $s \in R$ such that $\alpha(s) = r$ and $\alpha(a)rb = \alpha(as\alpha(b)) = 0$ as $\alpha^2 = 1_R$. Therefore $\alpha(a)Rb = 0$ and so using the reflexive property of R, we have $bR\alpha(a) = 0$. Thus R is right α-quasi reflexive.

(2) $\Rightarrow$ (1) Assume that R is right α-quasi reflexive. Let $a, b \in R$ such that $aRb = 0$. Since $\alpha^2 = 1_R$ so $0 = aRb = aR\alpha(\alpha(b))$ and using right α-quasi reflexive property of R, we have $\alpha(b)R\alpha(a) = 0$. Then for any $r \in R$, $\alpha(bra) = \alpha(b)\alpha(r)\alpha(a) = 0$, yielding $bra = 0$ as α is injective. Thus $bRa = 0$ and hence R is reflexive.

(1) $\Leftrightarrow$ (3) can be proved similarly. $\square$

Remark 1.10. (1) For a reduced ring R, the ring S (as defined in Proposition 1.3) is reflexive by [11, Proposition 2.5(2)(ii)] as reduced rings are semiprime. Moreover, $\alpha^2 = 1_S$ and so by Proposition 1.9, we can directly conclude that S is α-quasi reflexive.

(2) The conclusion of Proposition 1.3 remains true if we replace the condition "R is a reduced ring" by "R is a semiprime ring" as for a semiprime ring R, the ring S (as defined in Proposition 1.3) is reflexive by [11, Proposition 2.5(2)(ii)] and $\alpha^2 = 1_S$ implies S is α-quasi reflexive by Proposition 1.9. By the same argument, we can show that for a commutative ring R, S is α-quasi reflexive.

Corollary 1.11. *For a commutative or a semiprime ring R, the ring*

$$S = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} : a, b \in R \right\}$$

is α-quasi reflexive where $\alpha : S \to S$ is defined by

$$\alpha\left(\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\right) = \begin{pmatrix} a & -b \\ 0 & a \end{pmatrix}.$$

2. Properties and extensions

In this section, we study ring theoretic properties and extensions related to the right version of α-quasi reflexive rings.

For a nonempty subset S of a ring R, the *right annihilator* of S in R is denoted and defined by $r_R(S) = \{r \in R : Sr = 0\}$. The *left annihilator* is defined analogously and denoted by $\ell_R(S)$.

Proposition 2.1. *For a ring R with an endomorphism α, the following are equivalent:*

(1) *R is right α-quasi reflexive.*
(2) *For each $a \in R$, $\alpha^{-1}(r_R(aR)) = \ell_R(R\alpha(a))$.*
(3) *For any nonempty subsets A, B of R, $AR\alpha(B) = 0$ if and only if $BR\alpha(A) = 0$.*
(4) *For all right ideals I, J of R, $I\alpha(J) = 0$ if and only if $J\alpha(I) = 0$.*
(5) *For all ideals I, J of R, $I\alpha(J) = 0$ if and only if $J\alpha(I) = 0$.*

Proof. (1) $\Rightarrow$ (2) Let R be right α-quasi reflexive. To prove $\alpha^{-1}(r_R(aR)) = \ell_R(R\alpha(a))$, assume that $b \in \alpha^{-1}(r_R(aR))$. This gives $\alpha(b) \in r_R(aR)$ and so $aR\alpha(b) = 0$, entailing $bR\alpha(a) = 0$ as R is right α-quasi reflexive. This implies $b \in \ell_R(R\alpha(a))$ and so $\alpha^{-1}(r_R(aR)) \subseteq \ell_R(R\alpha(a))$.

Again let $b \in \ell_R(R\alpha(a))$. This gives $bR\alpha(a) = 0$. Since R is right α-quasi reflexive, therefore $aR\alpha(b) = 0$, entailing $\alpha(b) \in r_R(aR)$ and so $b \in \alpha^{-1}(r_R(aR))$. Thus $\ell_R(R\alpha(a)) \subseteq \alpha^{-1}(r_R(aR))$ and hence $\alpha^{-1}(r_R(aR)) = \ell_R(R\alpha(a))$.

(2) $\Rightarrow$ (3) Assume that (2) holds. Let A, B be nonempty subsets of R such that $AR\alpha(B) = 0$. Then for all $a \in A$ and $b \in B$, $aR\alpha(b) = 0$. Clearly, $b \in \alpha^{-1}(r_R(aR)) = \ell_R(R\alpha(a))$ and so $bR\alpha(a) = 0$ for all $a \in A$, $b \in B$. Thus we have

$$BR\alpha(A) = \sum_{\substack{a\in A, b\in B \\ \text{(finite)}}} bR\alpha(a) = 0.$$

Interchanging the roles of A and B, we obtain $BR\alpha(A) = 0$ implies $AR\alpha(B) = 0$.

(3) $\Rightarrow$ (4) Assume that (3) holds. Let I, J be two right ideals of R such that $I\alpha(J) = 0$. Then $IR = I$ and $JR = J$. Thus $0 = I\alpha(J) = IR\alpha(J)$ and so by assumption, we have $0 = JR\alpha(I) = J\alpha(I)$. Interchanging the roles of I and J, we see that $J\alpha(I) = 0$ implies $I\alpha(J) = 0$.

(4) $\Rightarrow$ (5) is straightforward.

(5) $\Rightarrow$ (1) Assume that (5) holds. Let $a, b \in R$ such that $aR\alpha(b) = 0$. Then RaR and RbR are ideals of R such that $RaR\alpha(RbR) = 0$. By assumption, $RbR\alpha(RaR) = 0$ and so $bR\alpha(a) \subseteq RbR\alpha(RaR) = 0$. Hence R is right α-quasi reflexive. $\square$

For a ring R and for $n \geq 2$, consider the following rings

$$D_n(R) = \left\{ \begin{pmatrix} a & a_{12} & \cdots & a_{1n} \\ 0 & a & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} : a, a_{ij} \in R \right\}$$

and

$$V_n(R) = \left\{ \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ 0 & a_1 & \cdots & a_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_1 \end{pmatrix} : a_i \in R \right\}$$

which are subrings of $M_n(R)$. Note that $R[x]/(x^n) \cong V_n(R)$, where (x^n) is the ideal of $R[x]$ generated by x^n.

For simplicity, we use $(a_1, a_2, \ldots, a_n) \in V_n(R)$ to denote

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ 0 & a_1 & \cdots & a_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_1 \end{pmatrix}.$$

For a ring R with an endomorphism α, the correspondence $\big(a_{ij}\big) \overset{\bar{\alpha}}{\mapsto} \big(\alpha(a_{ij})\big)$ induce endomorphisms of $M_n(R)$, $U_n(R)$, $D_n(R)$ and $V_n(R)$.

Proposition 2.2. *For a ring R with an endomorphism α, R is right α-quasi reflexive if and only if $M_n(R)$ is right $\bar{\alpha}$-quasi reflexive for $n \geq 2$.*

Proof. Assume that $n \geq 2$. Let R be right α-quasi reflexive and let I, J be ideals in $M_n(R)$ such that $I\bar{\alpha}(J) = 0$. Then there exist ideals A, B of R such that $I = M_n(A)$ and $J = M_n(B)$. This gives $0 = I\bar{\alpha}(J) = M_n(A)\bar{\alpha}(M_n(B)) = M_n(A)M_n(\alpha(B)) = M_n(A\alpha(B))$, entailing $A\alpha(B) = 0$. Since R is right α-quasi reflexive, therefore by Proposition 2.1(5), $B\alpha(A) = 0$ and so

$$J\bar{\alpha}(I) = M_n(B)\bar{\alpha}(M_n(A)) = M_n(B\alpha(A)) = 0.$$

Thus $M_n(R)$ is right $\bar{\alpha}$-quasi reflexive by Proposition 2.1(5). Conversely, assume that $M_n(R)$ is right $\bar{\alpha}$-quasi reflexive. Let A, B be ideals in R such that $A\alpha(B) = 0$. Then $M_n(A)$ and $M_n(B)$ are ideals in $M_n(R)$ such that $M_n(A)\bar{\alpha}(M_n(B)) = M_n(A\alpha(B)) = 0$ and so $M_n(B)\bar{\alpha}(M_n(A)) = 0$ by assumption. This gives $B\alpha(A) = 0$ and so R is right α-quasi reflexive by Proposition 2.1(5). $\square$

Proposition 2.3. *For any ring R with an endomorphism α such that $\alpha(1) = 1$, we have*
(1) *$U_n(R)$ is not right $\bar{\alpha}$-quasi reflexive for $n \geq 2$.*
(2) *$D_n(R)$ is not right $\bar{\alpha}$-quasi reflexive for $n \geq 3$.*

Proof. (1) Assume that $n \geq 2$. For $A = E_{12}$, $B = E_{11} \in U_n(R)$, it is clear that $AU_n(R)\bar{\alpha}(B) = 0$, however, $B\bar{\alpha}(A) = A \neq 0$, entailing $BU_n(R)\bar{\alpha}(A) \neq 0$ and so $U_n(R)$ is not right $\bar{\alpha}$-quasi reflexive.

(2) Assume that $n \geq 3$. For $A = E_{23}$, $B = E_{12} \in D_n(R)$, we have $AD_n(R)\bar{\alpha}(B) = 0$, however, $B\bar{\alpha}(A) = E_{13} \neq 0$, entailing $BD_n(R)\bar{\alpha}(A) \neq 0$ and so $D_n(R)$ is not right $\bar{\alpha}$-quasi reflexive. $\square$

Remark 2.4. (1) Let A be a ring and α an endomorphism of A such that $\alpha(1) = 1$. Consider the rings $R = M_2(A)$ and $S = U_2(A)$. Clearly, S is a subring of R. Note that R is right $\bar{\alpha}$-quasi reflexive by Proposition 2.2, however, S is not right $\bar{\alpha}$-quasi reflexive by Proposition 2.3(1). This shows that the class of right α-quasi reflexive rings is not closed under subrings.

(2) Recall that a ring R is called *directly finite* if for $a, b \in R$, $ab = 1$ implies $ba = 1$. Abelian rings are directly finite by [13, Lemma 3.4]. Following [15, Theorem 1.0], there exists a domain D for which $M_2(D)$ is not directly finite. For a given monomorphism α of D, D is right α-quasi reflexive by Remark 1.2(2) and so by Proposition 2.2, $M_2(D)$ is right $\bar{\alpha}$-quasi reflexive. This shows that a right α-quasi reflexive ring need not be directly finite.

Proposition 2.5. *For a semiprime ring R with an endomorphism α, R is right α-quasi reflexive if and only if $V_n(R)$ is right $\bar{\alpha}$-quasi reflexive for $n \geq 2$.*

Proof. Note that for a semiprime ring R, $aRb = 0$ if and only if $aRbRb = 0$ for all $a, b \in R$, by [11, Proposition 2.5(2)(i)]. We use this fact freely without reference.

Assume that $n \geq 2$. Let R be right α-quasi reflexive and let $A = (a_1, a_2, \ldots, a_n)$, $B = (b_1, b_2, \ldots, b_n) \in V_n(R)$ such that $AV_n(R)\bar{\alpha}(B) = 0$. Then for any $r \in R$, $A(r, 0, \ldots, 0)\bar{\alpha}(B) = 0$. Thus we have the following equations.

$$a_1 r\alpha(b_1) = 0. \tag{4}$$

$$a_1 r\alpha(b_2) + a_2 r\alpha(b_1) = 0. \tag{5}$$

$$a_1 r\alpha(b_3) + a_2 r\alpha(b_2) + a_3 r\alpha(b_1) = 0. \tag{6}$$

$$\vdots$$

$$a_1 r\alpha(b_k) + a_2 r\alpha(b_{k-1}) + \cdots + a_{k-1} r\alpha(b_2) + a_k r\alpha(b_1) = 0. \tag{7}$$

$$\vdots$$

$$a_1 r\alpha(b_n) + a_2 r\alpha(b_{n-1}) + \cdots + a_{n-1} r\alpha(b_2) + a_n r\alpha(b_1) = 0. \tag{8}$$

From (4), we obtain

$$a_1 R\alpha(b_1) = 0. \tag{9}$$

Multiplying (5) by $s\alpha(b_1)$ from right for any $s \in R$ and using (9), we get

$$a_2 R\alpha(b_1) = 0. \tag{10}$$

Therefore (5) becomes

$$a_1 R\alpha(b_2) = 0. \tag{11}$$

Thus we have

$$a_i R\alpha(b_j) = 0 \text{ for all } 2 \le i + j \le 3. \tag{12}$$

Assume that

$$a_i R\alpha(b_j) = 0 \text{ for all } 2 \le i + j \le k. \tag{13}$$

Multiplying (7) from right by $s_1\alpha(b_1), s_2\alpha(b_2), \ldots, s_{k-1}\alpha(b_{k-1})$ for any $s_1, s_2, \ldots, s_{k-1} \in R$, in turn, and using (13), we obtain

$$a_i R\alpha(b_j) = 0 \text{ for all } i + j = k. \tag{14}$$

By induction, we have

$$a_i R\alpha(b_j) = 0 \text{ for all } 2 \le i + j \le n + 1. \tag{15}$$

Since R is right α-quasi reflexive, therefore

$$b_j R\alpha(a_i) = 0 \text{ for all } 2 \le i + j \le n + 1$$

and so $BV_n(R)\bar{\alpha}(A) = 0$. Thus $V_n(R)$ is right $\bar{\alpha}$-quasi reflexive.

Conversely, assume that $V_n(R)$ is right $\bar{\alpha}$-quasi reflexive. Let $a, b \in R$ such that $aR\alpha(b) = 0$. Then $A = (a, 0, \ldots, 0)$, $B = (b, 0, \ldots, 0) \in V_n(R)$ such that $AV_n(R)\bar{\alpha}(B) = 0$. By assumption, $BV_n(R)\bar{\alpha}(A) = 0$ and so $bR\alpha(a) = 0$. Therefore R is right α-quasi reflexive. □

Remark 2.6. The condition "R is a semiprime ring" in Proposition 2.5 is not superfluous, i.e., for a right α-quasi reflexive ring R, $V_n(R)$ need not be right $\bar{\alpha}$-quasi reflexive for $n \ge 2$ by Example 2.15 to follow.

For a ring R and an (R, R)-bimodule M, the *trivial extension* of R by M is the ring $T(R, M) = R \oplus M$ with usual addition and the following multiplication:

$$(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + m_1r_2).$$

This ring is isomorphic to the ring of all matrices $\left(\begin{smallmatrix} r & m \\ 0 & r \end{smallmatrix}\right)$where $r \in R$ and $m \in M$, and the usual matrix operations are used.

Corollary 2.7. *For a semiprime ring R with an endomorphism α, the following are equivalent:*

(1) *R is right α-quasi reflexive.*
(2) *$T(R, R)$ is right $\bar{\alpha}$-quasi reflexive.*
(3) *$R[x]/(x^n)$ is right $\bar{\alpha}$-quasi reflexive for $n \ge 2$.*

For an endomorphism α of a ring R and an ideal I of R with $\alpha(I) \subseteq I$, the mapping $\bar{\alpha} : R/I \to R/I$ defined by $\bar{\alpha}(r + I) = \alpha(r) + I$ for $r \in R$, induces an endomorphism of R/I. The class of right α-quasi reflexive rings is not closed under homomorphic images by the help of [11, Example 2.8].

Proposition 2.8. *For a ring R with an endomorphism α and an ideal I of R with $\alpha(I) \subseteq I$, if R/I is right $\bar{\alpha}$-quasi reflexive and I is α-rigid as a ring (possibly without unity), then R is right α-quasi reflexive.*

Proof. Assume that I is α-rigid as a ring and R/I is right $\bar{\alpha}$-quasi reflexive. Let $a, b \in R$ such that $aR\alpha(b) = 0$. Then $(a + I)\ R/I\ \bar{\alpha}(b + I) = I$ and so $bR\alpha(a) \subseteq I$ as R/I is right $\bar{\alpha}$-quasi reflexive. Thus we have

$$bR\alpha(a)aR\alpha(bR\alpha(a)aR) = (bR\alpha(a))aR\alpha(b)\alpha(R\alpha(a)aR) = 0.$$

Clearly, $bR\alpha(a)aR \subseteq I$ and since I is α-rigid, therefore $bR\alpha(a)aR = 0$, entailing $bR\alpha(a)a = 0$. Since α-rigid rings are reduced and reduced rings are reversible, therefore $bR\alpha(a)a = 0$ implies $abR\alpha(a) = 0$. Thus

$$bR\alpha(a)\alpha(bR\alpha(a)) = bR\alpha(abR\alpha(a)) = 0$$

and so $bR\alpha(a) = 0$ as I is α-rigid. Hence R is right α-quasi reflexive. $\square$

The condition "I is α-rigid as a ring (possibly without unity)" in Proposition 2.8 is not superfluous by the following example.

Example 2.9. The argument here is due to [12, Example 2.10]. Let F be a field. Consider a ring $R = U_2(F)$ and an endomorphism $\alpha : R \to R$ defined by

$$\alpha\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} a & -b \\ 0 & c \end{pmatrix}.$$

Then $AR\alpha(B) = 0$ where

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \in R.$$

However, $B\alpha(A) = -A$ and so $BR\alpha(A) \neq 0$. Therefore R is not right α-quasi reflexive. Consider an ideal I of R, given by

$$I = \left\{\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} : b \in F\right\}.$$

Clearly, I is not α-rigid as $0 \neq A \in I$ and $A\alpha(A) = 0$. The quotient ring R/I, given by

$$R/I = \left\{\begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix} + I : a, c \in F\right\}$$

is reduced and $\bar{\alpha}$ is the identity endomorphism of R/I, entailing R/I is right $\bar{\alpha}$-quasi reflexive.

Next we show that the "right α-quasi reflexive property" is preserved under isomorphisms.

Proposition 2.10. *Let R be a ring with an endomorphism α. Let S be a ring and let $\sigma : R \to S$ be an isomorphism. Then R is right α-quasi reflexive if and only if S is right $\bar{\alpha}$-quasi reflexive where $\bar{\alpha} = \sigma \circ \alpha \circ \sigma^{-1}$.*

Proof. Let R be right α-quasi reflexive. Let $x, y \in S$ such that $xS\bar{\alpha}(y) = 0$. Since $\sigma : R \to S$ is bijective so there exist $a, b \in R$ such that $\sigma(a) = x$, $\sigma(b) = y$. Then

$$\sigma(aR\alpha(b)) = \sigma(a)\sigma(R)\sigma(\alpha(\sigma^{-1}(\sigma(b)))) = xS\bar{\alpha}(y) = 0$$

and so $aR\alpha(b) = 0$ as σ is bijective. This gives $bR\alpha(a) = 0$ as R is right α-quasi reflexive and so $yS\bar{\alpha}(x) = 0$ by similar argument. Therefore S is right $\bar{\alpha}$-quasi reflexive. Converse can be proved similarly. □

Proposition 2.11. *Let R be a ring with an endomorphism α such that $\alpha(e) = e$ for $e^2 = e \in R$.*

(1) *If R is right α-quasi reflexive then so is eRe.*

(2) *If $e \in Z(R)$, then R is right α-quasi reflexive if and only if eR and $(1-e)R$ are right α-quasi reflexive.*

Proof. (1) Let R be right α-quasi reflexive and let $a, b \in eRe$ such that $a(eRe)\alpha(b) = 0$. Then $ae = a = ea$ and $be = b = eb$. Thus $aR\alpha(b) = a(eRe)\alpha(b) = 0$ as $\alpha(e) = e$. Since R is right α-quasi reflexive, therefore $b(eRe)\alpha(a) = bR\alpha(a) = 0$ by similar argument. Hence eRe is right α-quasi reflexive.

(2) Necessity is clear from (1). Let $e \in Z(R)$. Assume that eR and $(1-e)R$ are right α-quasi reflexive. Let $a, b \in R$ such that $aR\alpha(b) = 0$. Since e is central idempotent in R and $\alpha(e) = e$, therefore $ea(eR)\alpha(eb) = 0$ and $(1-e)a((1-e)R)\alpha((1-e)b) = 0$. Since eR is right α-quasi reflexive so $ebR\alpha(a) = eb(eR)\alpha(ea) = 0$. Similarly, $(1-e)R$ is right α-quasi reflexive implies that $(1-e)bR\alpha(a) = 0$. Thus $bR\alpha(a) = ebR\alpha(a) + (1-e)bR\alpha(a) = 0$ and hence R is right α-quasi reflexive. □

Remark 2.12. Let $\{R_\lambda : \lambda \in \Lambda\}$ be a class of rings such that for each $\lambda \in \Lambda$, α_λ is an endomorphism of R_λ. Then the mapping $\bar{\alpha} : \prod_{\lambda\in\Lambda} R_\lambda \to \prod_{\lambda\in\Lambda} R_\lambda$ defined by $\bar{\alpha}((a_\lambda)) = (\alpha_\lambda(a_\lambda))$ induces an endomorphism of the direct product $\prod_{\lambda\in\Lambda} R_\lambda$. It is easy to prove that $\prod_{\lambda\in\Lambda} R_\lambda$ is right $\bar{\alpha}$-quasi reflexive if and only if each R_λ is right α_λ-quasi reflexive.

Due to Hirano [6], a ring R is called *quasi-Armendariz* if for $f(x) = \sum_{i=0}^{m} a_i x^i$, $g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$, $f(x)R[x]g(x) = 0$ implies $a_i R b_j = 0$ for all i, j.

For a ring R with an endomorphism α, the mapping $\bar{\alpha} : R[x] \to R[x]$ defined by

$$\bar{\alpha}\left(\sum_{i=0}^{n} a_i x^i\right) = \sum_{i=0}^{n} \alpha(a_i) x^i$$

induces an endomorphism of $R[x]$.

Proposition 2.13. *For an endomorphism α of a quasi-Armendariz ring R, the following are equivalent:*

(1) *R is right α-quasi reflexive.*

(2) *$R[x]$ is right $\bar{\alpha}$-quasi reflexive.*

Proof. (1) ⇒ (2) Let R be right α-quasi reflexive. Let $f(x) = \sum_{i=0}^{m} a_i x^i$, $g(x) = \sum_{j=0}^{n} b_j x^j$ in $R[x]$ such that $f(x)R[x]\bar{\alpha}(g(x)) = 0$. Since R is quasi-Armendariz, therefore $a_i R\alpha(b_j) = 0$ and so $b_j R\alpha(a_i) = 0$ for all i, j as R is right α-quasi reflexive. This gives $g(x)R[x]\bar{\alpha}(f(x)) = 0$ and so $R[x]$ is right $\bar{\alpha}$-quasi reflexive.

(2) ⇒ (1) Assume that $R[x]$ is right $\bar{\alpha}$-quasi reflexive. Let $a, b \in R$ such that $aR\alpha(b) = 0$. Then $aR[x]\bar{\alpha}(b) = aR[x]\alpha(b) = 0$ by [6, Lemma 2.1]. By assumption, $bR[x]\bar{\alpha}(a) = 0$ and so $bR\alpha(a) = 0$. Therefore R is right α-quasi reflexive. □

The condition "R is a quasi-Armendariz ring" in Proposition 2.13 is not superfluous, i.e., for a right α-quasi reflexive ring R, $R[x]$ need not be right $\bar{\alpha}$-quasi reflexive by the following example.

Example 2.14. We refer to the ring in [8, Example 2.8]. Let $A = \mathbb{Z}_2\{a_0, a_1, a_2, b_0, b_1, b_2, c\}$ be the free algebra with zero constant terms in noncommuting indeterminates a_0, a_1, a_2, b_0, b_1, b_2, c over $\mathbb{Z}_2$. Note that A is a ring without unity and consider an ideal of the ring $\mathbb{Z}_2 + A$, say I, generated by

$$\begin{aligned}
&a_0^2, b_0^2, a_2^2, b_2^2, a_0b_0, b_0a_0, a_2b_2, b_2a_2, a_0ra_0,\\
&b_0rb_0, a_0rb_0, b_0ra_0, a_2ra_2, b_2rb_2, a_2rb_2, b_2ra_2,\\
&a_0b_1 + a_1b_0, b_0a_1 + b_1a_0, a_1b_2 + a_2b_1, b_1a_2 + b_2a_1,\\
&a_0a_1 + a_1a_0, b_0b_1 + b_1b_0, a_1a_2 + a_2a_1,\\
&b_1b_2 + b_2b_1, a_0b_2 + a_1b_1 + a_2b_0, b_0a_2 + b_1a_1 + b_2a_0,\\
&a_0a_2 + a_1^2 + a_2a_0, b_0b_2 + b_1^2 + b_2b_0,\\
&(a_0 + a_1 + a_2)r(b_0 + b_1 + b_2), (b_0 + b_1 + b_2)r(a_0 + a_1 + a_2),\\
&(a_0 + a_1 + a_2)r(a_0 + a_1 + a_2),\\
&(b_0 + b_1 + b_2)r(b_0 + b_1 + b_2), r_1r_2r_3r_4,
\end{aligned}$$

where $r, r_1, r_2, r_3, r_4 \in A$. Then clearly $A^4 \subseteq I$. Let $R = (\mathbb{Z}_2 + A)/I$. Let σ be an automorphism of $\mathbb{Z}_2 + A$ defined by

$$a_0, a_1, a_2, b_0, b_1, b_2, c \mapsto b_0, b_1, b_2, a_0, a_1, a_2, c.$$

Since $\sigma(I) \subseteq I$, we obtain an endomorphism α of R such that $\alpha(s + I) = \sigma(s) + I$ for $s \in \mathbb{Z}_2 + A$.

We call each product of the indeterminates $a_0, a_1, a_2, b_0, b_1, b_2, c$ a monomial and a monomial of degree n means a product of exactly n number of indeterminates. Let H_n be the set of all linear combinations of monomials of degree n over $\mathbb{Z}_2$. Note that H_n is finite for any n and that the ideal I of R is homogeneous (i.e., if $\sum_{i=1}^{s} r_i \in I$ with $r_i \in H_i$ then every $r_i \in I$). We adopt the method used in [9, Example 2.1] to show that R is right α-quasi reflexive.

Claim 1. *If $f_1\sigma(g_1) \in I$ for $f_1, g_1 \in H_1$ then $g_1\sigma(f_1) \in I$.*

Proof. Let $f_1, g_1 \in H_1$ such that $f_1\sigma(g_1) \in I$. By the definition of I, we have the following cases:

$$\begin{gathered}
(f_1 = a_0, g_1 = a_0), (f_1 = a_0, g_1 = b_0), (f_1 = b_0, g_1 = b_0), (f_1 = b_0, g_1 = a_0),\\
(f_1 = a_2, g_1 = a_2), (f_1 = a_2, g_1 = b_2), (f_1 = b_2, g_1 = b_2), (f_1 = b_2, g_1 = a_2),\\
(f_1 = a_0 + a_1 + a_2, g_1 = a_0 + a_1 + a_2), (f_1 = a_0 + a_1 + a_2, g_1 = b_0 + b_1 + b_2),\\
(f_1 = b_0 + b_1 + b_2, g_1 = b_0 + b_1 + b_2), (f_1 = b_0 + b_1 + b_2, g_1 = a_0 + a_1 + a_2).
\end{gathered}$$

So we obtain the result, using the definition of I and σ.

Claim 2. *If $f\sigma(g) \in I$ for $f, g \in A$ then $g\sigma(f) \in I$.*

Proof. Let $f, g \in A$ such that $f\sigma(g) \in I$. We may write $f = f_1 + f_2 + f_3 + f_4$, $g = g_1 + g_2 + g_3 + g_4$ for some $f_i, g_i \in H_i$ (for $i = 1, 2, 3$) and some $f_4, g_4 \in I$ as $H_i \subseteq I$ for $i \geq 4$. Then $f\sigma(g) = f_1\sigma(g_1) + f_1\sigma(g_2) + f_2\sigma(g_1) + h$ with $h \in I$. Thus $f\sigma(g) \in I$ implies $f_1\sigma(g_1) + f_1\sigma(g_2) + f_2\sigma(g_1) \in I$. Since I is homogeneous, therefore $f_1\sigma(g_1) \in I$, $f_1\sigma(g_2) + f_2\sigma(g_1) \in I$. From $f_1\sigma(g_1) \in I$, we have $g_1\sigma(f_1) \in I$ by Claim 1. We show that $g_1\sigma(f_2) + g_2\sigma(f_1) \in I$. From $f_1\sigma(g_2) + f_2\sigma(g_1) \in I$, we have the following cases:

$$(f_1 = a_0, g_1 = a_0), (f_1 = a_0, g_1 = b_0), (f_1 = b_0, g_1 = b_0), (f_1 = b_0, g_1 = a_0),$$
$$(f_1 = a_2, g_1 = a_2), (f_1 = a_2, g_1 = b_2), (f_1 = b_2, g_1 = b_2), (f_1 = b_2, g_1 = a_2),$$
$$(f_1 = a_0 + a_1 + a_2, g_1 = a_0 + a_1 + a_2), (f_1 = a_0 + a_1 + a_2, g_1 = b_0 + b_1 + b_2),$$
$$(f_1 = b_0 + b_1 + b_2, g_1 = b_0 + b_1 + b_2), (f_1 = b_0 + b_1 + b_2, g_1 = a_0 + a_1 + a_2).$$

If $f_2, g_2 \in I$ then clearly $g_1\sigma(f_2) + g_2\sigma(f_1) \in I$. So we consider other cases of f_2 and g_2. When $f_1 = a_0, g_1 = a_0$, we may obtain the following cases:

$$(f_2 \in I, g_2 = a_0t), (f_2 \in I, g_2 = ta_0), (f_2 \in I, g_2 = b_0t), (f_2 \in I, g_2 = tb_0),$$
$$(f_2 = a_0s, g_2 \in I), (f_2 = sa_0, g_2 \in I), (f_2 = b_0s, g_2 \in I), (f_2 = sb_0, g_2 \in I),$$
$$(f_2 = a_0s, g_2 = a_0t), (f_2 = a_0s, g_2 = ta_0), (f_2 = a_0s, g_2 = b_0t), (f_2 = a_0s, g_2 = tb_0),$$
$$(f_2 = sa_0, g_2 = a_0t), (f_2 = sa_0, g_2 = ta_0), (f_2 = sa_0, g_2 = b_0t), (f_2 = sa_0, g_2 = tb_0),$$
$$(f_2 = b_0s, g_2 = a_0t), (f_2 = b_0s, g_2 = ta_0), (f_2 = b_0s, g_2 = b_0t), (f_2 = b_0s, g_2 = tb_0),$$
$$(f_2 = sb_0, g_2 = a_0t), (f_2 = sb_0, g_2 = ta_0), (f_2 = sb_0, g_2 = b_0t), (f_2 = sb_0, g_2 = tb_0),$$

where $s, t \in H_1$. Then $g_1\sigma(f_2) + g_2\sigma(f_1) = a_0\sigma(f_2) + g_2b_0 \in I$.

The computations for other cases are similar. Thus $g\sigma(f) = g_1\sigma(f_1) + g_1\sigma(f_2) + g_2\sigma(f_1) + k$ with $k \in I$, is also contained in I.

Claim 3. *If $g\sigma(h) \in I$ for $g, h \in \mathbb{Z}_2 + A$ then $h\sigma(g) \in I$.*

Proof. Let $g, h \in \mathbb{Z}_2 + A$ such that $g\sigma(h) \in I$. We may write $g = k + g'$, $h = \ell + h'$ for some $k, \ell \in \mathbb{Z}_2$ and some $g', h' \in A$. Since $g\sigma(h) = k\ell + k\sigma(h') + g'\ell + g'\sigma(h') \in I$, therefore $k = 0$ or $\ell = 0$. Assume that $k = 0$. Then $g'l + g'\sigma(h') \in I$ and so $g', g'\sigma(h') \in I$ as I is homogeneous and $\ell \in \mathbb{Z}_2$. Therefore by Claim 2, $h'\sigma(g') \in I$ and consequently $h\sigma(g) \in I$. For the case of $\ell = 0$, we obtain $h\sigma(g) \in I$ similarly.

Claim 4. *If $g(\mathbb{Z}_2 + A)\sigma(h) \in I$ for $g, h \in \mathbb{Z}_2 + A$ then $h(\mathbb{Z}_2 + A)\sigma(g) \in I$.*

Proof. Let $g, h \in \mathbb{Z}_2 + A$ such that $g(\mathbb{Z}_2 + A)\sigma(h) \in I$. Clearly, $g\sigma(h) \in I$ and so for any $r \in \mathbb{Z}_2 + A$, we have $g\sigma(hr) \in I$. By Claim 3, $hr\sigma(g) \in I$. Since $r \in \mathbb{Z}_2 + A$ is arbitrary, therefore $h(\mathbb{Z}_2 + A)\sigma(g) \in I$.

Claim 5. *R is right α-quasi reflexive.*

Proof. Let $g + I, h + I \in R$ such that $(g + I)R\alpha(h + I) = I$. Then $g(\mathbb{Z}_2 + A)\sigma(h) \in I$ and so by Claim 4, $h(\mathbb{Z}_2 + A)\sigma(g) \in I$, entailing $(h + I)R\alpha(g + I) = I$. Therefore R is right α-quasi reflexive.

Consider $R[x] \cong (\mathbb{Z}_2 + A)[x]/I[x]$. For simplicity, we identify the elements of $\mathbb{Z}_2 + A$ with their images in R. For $f(x) = a_0 + a_1x + a_2x^2$, $g(x) = b_0c + b_1cx + b_2cx^2 \in R[x]$ and for any $r = k + h \in R$, where $k \in \mathbb{Z}_2$ and $h \in A$, we have

$$\begin{aligned} f(x)r\bar{\alpha}(g(x)) &= (a_0 + a_1x + a_2x^2)r(a_0c + a_1cx + a_2cx^2) \\ &= (a_0a_1 + a_1a_0)kcx + (a_0a_2 + a_1^2 + a_2a_0)kcx^2 + (a_1a_2 + a_2a_1)kcx^3 \\ &\quad + (a_0ha_1 + a_1ha_0)cx + (a_0ha_2 + a_1ha_1 + a_2ha_0)cx^2 \\ &\quad + (a_1ha_2 + a_2ha_1)cx^3 \\ &= 0. \end{aligned}$$

Since $r \in R$ is arbitrary, therefore $f(x)R\bar{\alpha}(g(x)) = 0$ and so by [6, Lemma 2.1], we have $f(x)R[x]\bar{\alpha}(g(x)) = 0$. However, $g(x)\bar{\alpha}(f(x)) \neq 0$ as $b_0cb_1 + b_1cb_0 \neq 0$ and so $g(x)R[x]\bar{\alpha}(f(x)) \neq 0$. Therefore $R[x]$ is not right $\bar{\alpha}$-quasi reflexive.

Next we show that for a right α-quasi reflexive ring R, $V_n(R)$ need not be right $\bar{\alpha}$-quasi reflexive for $n \geq 2$.

Example 2.15. Consider the ring in Example 2.14, i.e., $R = (\mathbb{Z}_2 + A)/I$ with the endomorphism α where A, I and α are as defined in Example 2.14. Then R is right α-quasi reflexive. For simplicity, we identify the elements of $\mathbb{Z}_2 + A$ with their images in R. Assume that $n \geq 2$. For $A = (a_0, 0, \ldots, 0, a_1)$, $B = (b_0c, 0, \ldots, 0, b_1c) \in V_n(R)$, we have $AV_n(R)\bar{\alpha}(B) = 0$ by applying arguments similar to those given in Example 1.4. However, $B\bar{\alpha}(A) \neq 0$ as $b_0cb_1 + b_1cb_0 \neq 0$, entailing $BV_n(R)\bar{\alpha}(A) \neq 0$ and so $V_n(R)$ is not right $\bar{\alpha}$-quasi reflexive.

A *regular* element in a ring R is any nonzero divisor. For a multiplicatively closed (m.c. for short) subset S of a ring R consisting of regular elements, we denote by RS^{-1} (resp., $S^{-1}R$), the right (resp., left) localization of R at S, which is also called the *right* (resp., *left*) *quotient* ring of R with respect to S. An m.c. subset S of a ring R is called *right* (resp., *left*) *Ore* if for each $r \in R$ and $s \in S$, there exist $r_1 \in R$ and $s_1 \in S$ such that $rs_1 = sr_1$ (resp., $s_1r = r_1s$), i.e., $rS \cap sR \neq \emptyset$ (resp., $Sr \cap Rs \neq \emptyset$). Following [5, Theorem 6.2], an m.c. subset S of a ring R consisting of regular elements is right (resp., left) Ore if and only if the right (resp., left) quotient ring of R with respect to S exists.

For an automorphism α of a ring R with $\alpha(S) \subseteq S$ where S is an m.c. subset of R consisting of regular elements, the mapping $\bar{\alpha} : RS^{-1} \to RS^{-1}$ defined by $\bar{\alpha}(rs^{-1}) = \alpha(r)\alpha(s)^{-1}$ for $r \in R$ and $s \in S$, induces an automorphism of RS^{-1}. The induced automorphism of $S^{-1}R$ is defined analogously.

Proposition 2.16. *For a right Ore subset S of a ring R consisting of regular elements and an automorphism α of R with $\alpha(S) \subseteq S$, if R is right α-quasi reflexive, then RS^{-1} is right $\bar{\alpha}$-quasi reflexive.*

Proof. Let R be right α-quasi reflexive and let $A = au^{-1}$, $B = bv^{-1} \in RS^{-1}$ be such that $A(RS^{-1})\bar{\alpha}(B) = 0$ where $a, b \in R$ and $u, v \in S$. Then $0 = A(RS^{-1})\bar{\alpha}(B) = a(RS^{-1})\alpha(b)\alpha(v)^{-1}$ as $u^{-1}(RS^{-1}) = RS^{-1}$. Then for any $rs^{-1} \in RS^{-1}$, we have $a(rs^{-1})\alpha(b)\alpha(v)^{-1} = 0$. For $\alpha(b) \in R$ and $s \in S$, there exist $b_1 \in R$ and $s_1 \in S$ such that $\alpha(b)s_1 = sb_1$ and $s^{-1}\alpha(b) = b_1s_1^{-1}$ as S is right Ore. This gives $0 = ar(s^{-1}\alpha(b))\alpha(v)^{-1} = arb_1s_1^{-1}\alpha(v)^{-1}$ for any $r \in R$ and so $aRb_1 = 0$. From $aRb_1 = 0$ and $\alpha(b)s_1 = sb_1$, we have $0 = arsb_1 = ar\alpha(b)s_1$ for any $r \in R$ and so $aR\alpha(b) = 0$. Again $v^{-1}(RS^{-1}) = RS^{-1}$ and so

$B(RS^{-1})\bar{\alpha}(A) = b(RS^{-1})\alpha(a)\alpha(u)^{-1}$. Consider $b(rs^{-1})\alpha(a)\alpha(u)^{-1}$ for any $rs^{-1} \in RS^{-1}$. Since $\alpha(a) \in R$ and $s \in S$, there exist $a_2 \in R$ and $s_2 \in S$ such that $\alpha(a)s_2 = sa_2$ and $s^{-1}\alpha(a) = a_2 s_2^{-1}$ by similar argument. Then $b(rs^{-1})\alpha(a)\alpha(u)^{-1} = (bra_2)s_2^{-1}\alpha(u)^{-1}$. Since α is an automorphism of R and $\alpha(S) \subseteq S$, therefore there exist $a_2' \in R$ and $s', s_2' \in S$ such that $\alpha(a_2') = a_2$, $\alpha(s') = s$ and $\alpha(s_2') = s_2$. Then $\alpha(a)s_2 = sa_2$ implies $as_2' = s'a_2'$ and so from $aR\alpha(b) = 0$, we have $0 = as_2'r\alpha(b) = s'a_2'r\alpha(b)$ for any $r \in R$ and so $a_2'R\alpha(b) = 0$. Since R is right α-quasi reflexive, therefore $0 = bR\alpha(a_2') = bRa_2$ and so $b(rs^{-1})\alpha(a)\alpha(u)^{-1} = (bra_2)s_2^{-1}\alpha(u)^{-1} = 0$ for any $rs^{-1} \in RS^{-1}$. Therefore $B(RS^{-1})\bar{\alpha}(A) = 0$ and hence RS^{-1} is right $\bar{\alpha}$-quasi reflexive. □

By applying arguments similar to those given in the proof of Proposition 2.16, we have the following.

Proposition 2.17. *For a left Ore subset S of a ring R consisting of regular elements and an automorphism α of R with $\alpha(S) \subseteq S$, if R is right α-quasi reflexive, then $S^{-1}R$ is right $\bar{\alpha}$-quasi reflexive.*

Proposition 2.18. *For an m.c. subset S of a ring R consisting of central regular elements and an endomorphism α of R with $\alpha(S) \subseteq S$ and $\alpha(1) = 1$, R is right α-quasi reflexive if and only if $S^{-1}R$ is right $\bar{\alpha}$-quasi reflexive.*

Proof. Necessity is clear from the proof of Proposition 2.16. For the sufficiency part, assume that $S^{-1}R$ is right $\bar{\alpha}$-quasi reflexive. Let $a, b \in R$ such that $aR\alpha(b) = 0$. Then $a(S^{-1}R)\bar{\alpha}(b) = 0$. By assumption, $b(S^{-1}R)\bar{\alpha}(a) = 0$ and so $bR\alpha(a) = 0$ as $\alpha(1) = 1$. Hence R is right α-quasi reflexive. □

The ring of *Laurent polynomials* in x over a ring R, consisting of all formal sums $\sum_{i=k}^{n} r_i x^i$ with usual addition and multiplication, where $r_i \in R$ and k, n are integers, is denoted by $R[x; x^{-1}]$.

For a ring R with an endomorphism α, $\bar{\alpha} : R[x; x^{-1}] \to R[x; x^{-1}]$ defined by

$$\bar{\alpha}\left(\sum_{i=k}^{n} r_i x^i\right) = \sum_{i=k}^{n} \alpha(r_i) x^i$$

induces an endomorphism of $R[x; x^{-1}]$.

Corollary 2.19. *For a ring R with an endomorphism α such that $\alpha(1) = 1$, $R[x]$ is right $\bar{\alpha}$-quasi reflexive if and only if $R[x; x^{-1}]$ is right $\bar{\alpha}$-quasi reflexive.*

Proof. Since $S = \{1, x, x^2, \ldots\}$ is an m.c. subset of $R[x]$ such that $R[x; x^{-1}] = S^{-1}R[x]$, it follows directly from Proposition 2.18. □

For an algebra R over a commutative ring S, the *Dorroh extension* [4] of R by S is the abelian group $D = R \oplus S$ with multiplication given by

$$(r_1, s_1)(r_2, s_2) = (r_1 r_2 + s_1 r_2 + s_2 r_1, s_1 s_2).$$

For an S-endomorphism α of R and the Dorroh extension D of R by S, $\bar{\alpha} : D \to D$ defined by $\bar{\alpha}((r, s)) = (\alpha(r), s)$, is an S-algebra endomorphism.

Proposition 2.20. *Let R be an algebra over a commutative ring S and α an S-endomorphism of R with $\alpha(1) = 1$. Then R is right α-quasi reflexive if and only if the Dorroh extension D of R by S is right $\bar{\alpha}$-quasi reflexive.*

Proof. Clearly, any $s \in S$ can be written as $s = s1 \in R$ and so $R = \{r + s : (r, s) \in D\}$. Let R be right α-quasi reflexive and let $(r_1, s_1), (r_2, s_2) \in D$ such that $(r_1, s_1)D\bar{\alpha}((r_2, s_2)) = 0$. Then for any $(r, s) \in D$, $(r_1, s_1)(r, s)(\alpha(r_2), s_2) = 0$. This gives $r_1r\alpha(r_2) + s_1r\alpha(r_2) + sr_1\alpha(r_2) + s_2r_1r + s_2s_1r + s_2sr_1 + s_1s\alpha(r_2) = 0$ and $s_1ss_2 = 0$. Thus $(r_1, s_1)(r, s)(\alpha(r_2), s_2) = 0$ is equivalent to $(r_1 + s_1)(r + s)(\alpha(r_2) + s_2) = 0$ and $s_1ss_2 = 0$. Therefore $(r_1 + s_1)R\alpha(r_2 + s_2) = 0$ as $\alpha(1) = 1$ and $s_1Ss_2 = 0$. Since R is right α-quasi reflexive and S is commutative, therefore $(r_2 + s_2)R\alpha(r_1 + s_1) = 0$ and $s_2Ss_1 = 0$ and so $(r_2, s_2)D\bar{\alpha}((r_1, s_1)) = 0$ by similar arguments. Therefore D is right $\bar{\alpha}$-quasi reflexive. Conversely, assume that D is right $\bar{\alpha}$-quasi reflexive. Clearly, $e = (1, 0) \in D$ satisfy $e^2 = e$ and $\bar{\alpha}(e) = e$. Also, $eDe \cong R$ and so by Propositions 2.10 and 2.11(1), R is right α-quasi reflexive. $\square$

The condition "$\alpha(1) = 1$" in Proposition 2.20 is not superfluous by the following example.

Example 2.21. Consider a ring $R = U_2(\mathbb{Z}_2)$. Let $\alpha : R \to R$ be an endomorphism defined by

$$\alpha\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}.$$

Then R is right α-quasi reflexive by Example 1.8(1). Since any ring can be regarded as a $\mathbb{Z}$-algebra so we consider the Dorroh extension D of R by $\mathbb{Z}$. For

$$A = \left(\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, 1\right) \text{ and } B = \left(\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, 0\right) \in D = R \oplus \mathbb{Z}$$

it is clear that $AD\bar{\alpha}(B) = 0$ but $B\bar{\alpha}(A) = B \neq 0$ and so $BD\bar{\alpha}(A) \neq 0$. Therefore D is not right $\bar{\alpha}$-quasi reflexive.

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